Introduction to exoplanetology

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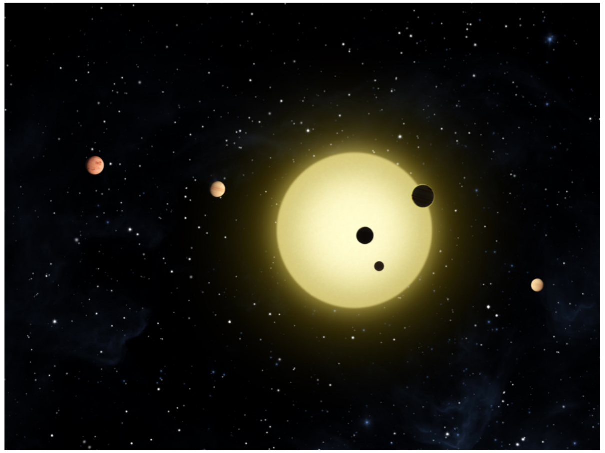
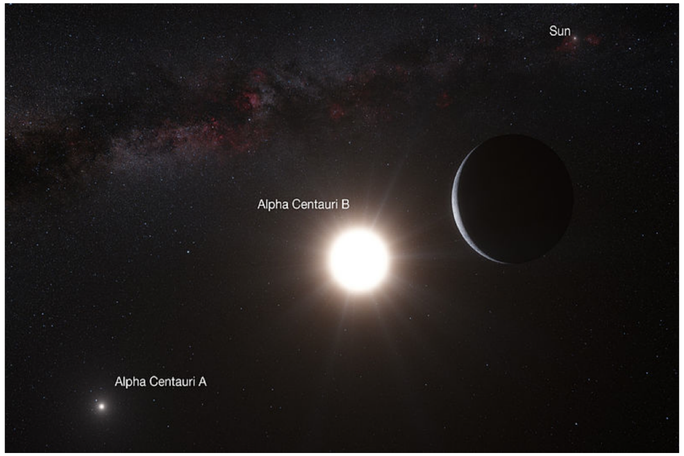
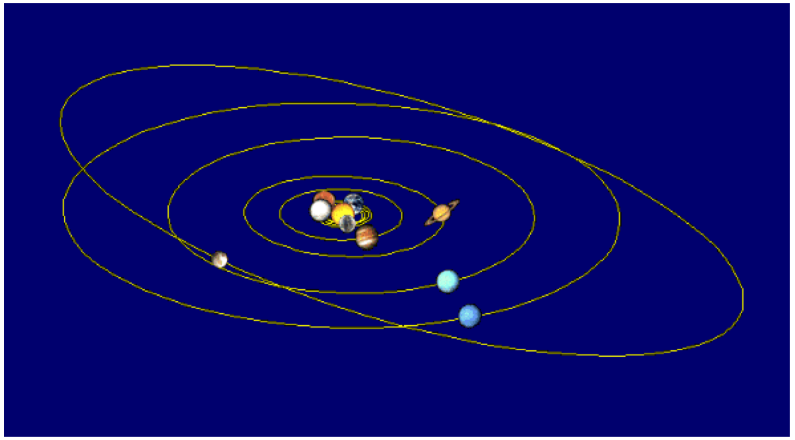


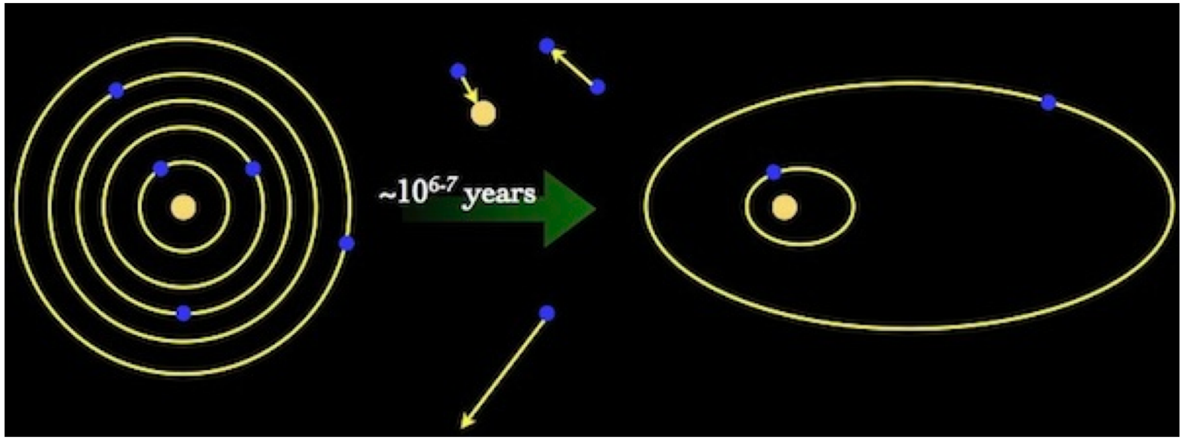


Master in Space Sciences – Academic year 2021-2022

Introduction to exoplanetology. II.

Planetary systems dynamics





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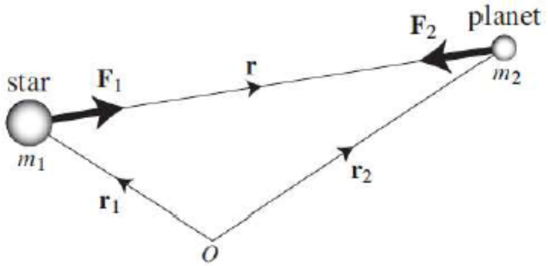
The two-body problems

Are assumed a star of mass m* and a planet of mass m_p. Their equations of motion in a random inertial reference system are:

$$\ddot{\boldsymbol{r}}_* = \frac{Gm_p(\boldsymbol{r}_p - \boldsymbol{r}_*)}{|\boldsymbol{r}_p - \boldsymbol{r}_*|^3}, \qquad \ddot{\boldsymbol{r}}_p = \frac{Gm_*(\boldsymbol{r}_* - \boldsymbol{r}_p)}{|\boldsymbol{r}_* - \boldsymbol{r}_p|^3}$$



Change of variables :
$$\boldsymbol{R} \equiv \frac{m_*\boldsymbol{r}_* + m_p\boldsymbol{r}_p}{m_* + m_p}, \qquad \boldsymbol{r} \equiv \boldsymbol{r}_p - \boldsymbol{r}_*$$

Vector of position of the center of mass

Vector of position of the Planet relative to the star

➡ $\ddot{\boldsymbol{R}} = 0$ Center of mass is in uniform motion

➡ $\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r}, \qquad M \equiv m_* + m_p$

Equation of motion for the planet relative to the star

= equation of relative motion of a test particle in orbit around a mass M

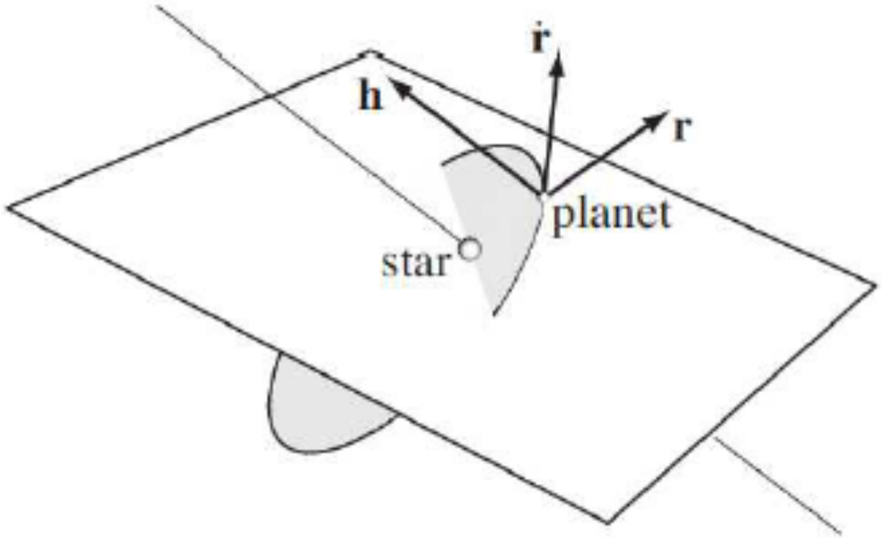
The two-body problem

$$\boldsymbol{h} = \boldsymbol{r} \wedge \dot{\boldsymbol{r}}$$

Orbital angular momentum

$$\dot{\boldsymbol{h}} = \dot{\boldsymbol{r}} \wedge \dot{\boldsymbol{r}} + \boldsymbol{r} \wedge \ddot{\boldsymbol{r}} = -\boldsymbol{r} \wedge \frac{GM}{r^3}\boldsymbol{r} = 0$$

Motion in a plane



The two-body problem

Representation in polar coordinates (*r*, ψ) system

$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_r \qquad d\hat{\boldsymbol{e}}_r/dt = \dot{\psi}\hat{\boldsymbol{e}}_\psi \qquad d\hat{\boldsymbol{e}}_\psi/dt = -\dot{\psi}\hat{\boldsymbol{e}}_r$$

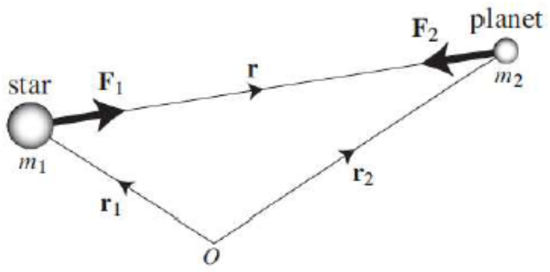
$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r} \Longrightarrow \ddot{r} - r\dot{\psi}^2 = -\frac{GM}{r^2}, \qquad 2\dot{r}\dot{\psi} + r\ddot{\psi} = 0$$

Orbital angular momentum

$2\dot{r}\dot{\psi} + r\ddot{\psi} = 0$: multiplication by r and integration $\Longrightarrow$ $\boxed{r^2\dot{\psi} = \text{constant} \quad = h}$

$$\ddot{r} - r\dot{\psi}^2 = -\frac{GM}{r^2} \Longrightarrow \ddot{r} - \frac{h^2}{r^3} = -\frac{GM}{r^2}$$

The two-body problem



Assuming polar coordinates r and ψ
r = star – planet distance
u=1/r, et ψ replaces *t* through $\frac{d}{dt} = \dot{\psi}\frac{d}{d\psi} = \frac{h}{r^2}\frac{d}{d\psi}$

$$\boxed{\frac{d^2u}{d\psi^2} + u = \frac{GM}{h^2}}$$

Non-homogeneous, second-order, linear differential equation

General solution:

phase term

$$u = \frac{1}{r} = \frac{GM}{h^2}[1 + e\cos(\psi - \varpi)]$$

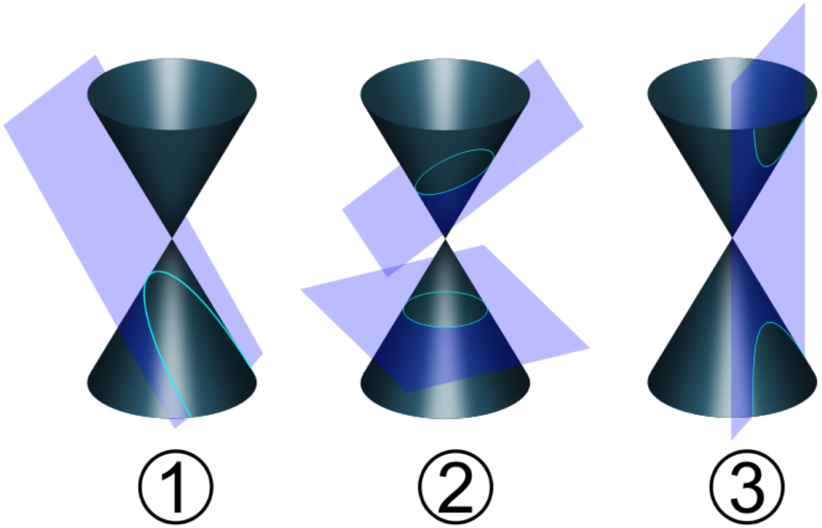
amplitude reference angle

We go back to r(ψ) :

$$\boxed{r = \frac{p}{1 + e\cos(\psi - \varpi)}}$$

p = h^2/GM = *semi-latus rectum*
e ≥ 0 = eccentricity
Equation of a conic section in polar coordinates

The two-body problem

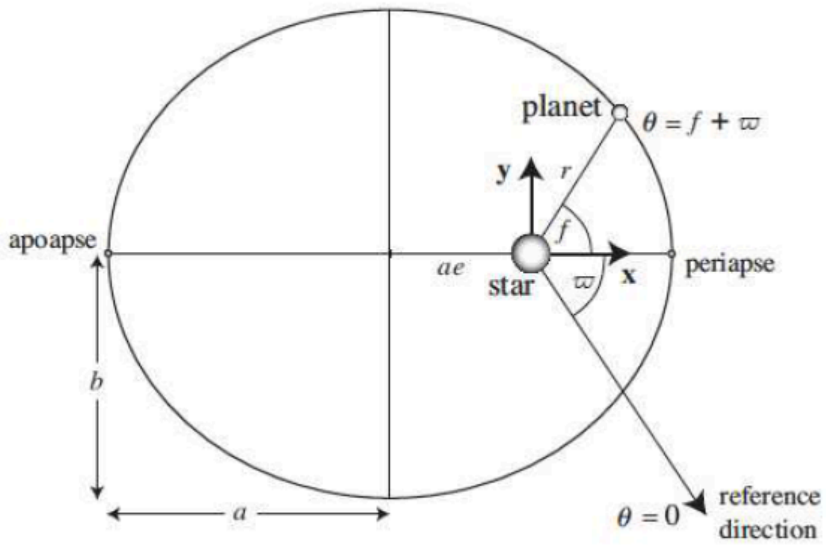


1 = parabola : $e = 1$
2 = ellipse : $e < 1$
3 = hyperbola : $e > 1$

1st law of Kepler

$e < 1$ -> ellipse with the star at one focus
$p = a(1-e^2)$, with a the semi-major axis
$p = h^2/GM$
$\mathbf{h = [GMa(1-e^2)]^{1/2}}$
Distance focus-centre = ae

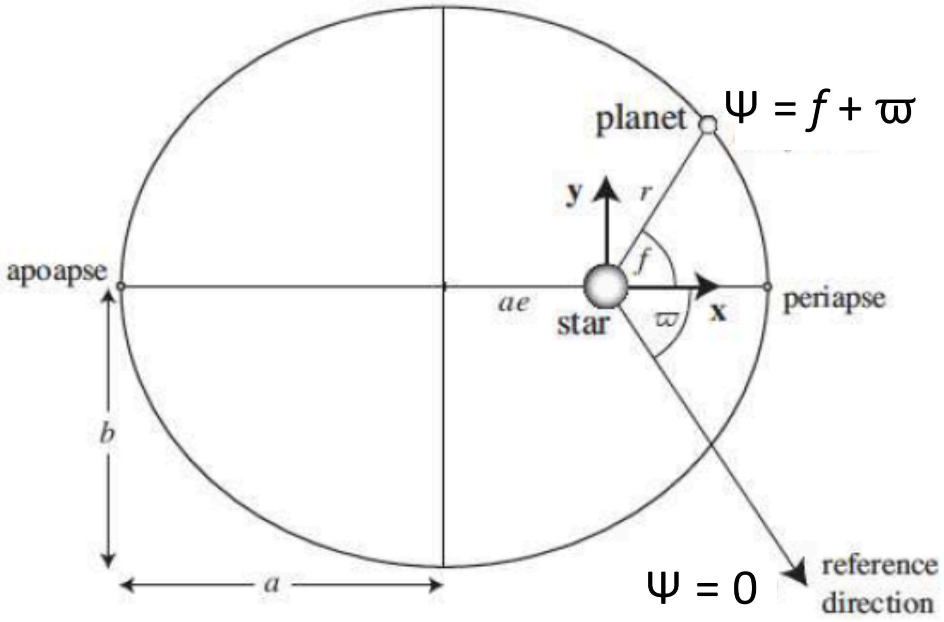
$$r = \frac{a(1-e^2)}{1 + e\cos f}$$



$$f = \psi - \varpi$$

True anomaly

The two-body problem



f = true anomaly

f = 0? -> pericentre
$\psi = \varpi$
$r = a(1-e)$
Ψ (= λ) = true longitude
ϖ = longitude of pericentre

f = 180° -> apocentre
$r = a(1+e)$

Area swept by the radius vector?

$$dA = \frac{1}{2} r^2 d\psi$$

$$\boxed{\dot{A} = \frac{1}{2} r^2 \dot{\psi} = \frac{1}{2} h}$$

→ **2nd law of Kepler**

The two-body problem

$$\dot{A} = \frac{1}{2}r^2\dot{\psi} = \frac{1}{2}h$$

Integration over a full orbit ➔ $A_{tot} = h\frac{P}{2}$ with P the orbital period

But the area of an ellipse is πab, avec $b^2 = a^2(1-e^2)$

➔ $$\pi a^2\sqrt{1-e^2} = h\frac{P}{2} = \sqrt{GMa(1-e^2)}\,\frac{P}{2}$$

➔ $$\boxed{P = 2\pi\frac{a^{\frac{3}{2}}}{\sqrt{GM}}}$$

3rd law of Kepler

The two-body problem

Orbital energy and velocity

Back to the equation of relative motion $\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r}$

Scalar product by $\dot{\boldsymbol{r}}$ $\dot{\boldsymbol{r}}.\ddot{\boldsymbol{r}} + \frac{GM}{r^3}\boldsymbol{r}.\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}.\ddot{\boldsymbol{r}} + \frac{GM}{r^2}\dot{r} = 0$

Integration ➔ $\frac{1}{2}v^2 - \frac{GM}{r} = C = \text{constant}$

$C = -\frac{GM}{2a}$ ➔ **Orbital energy does not depend on *e***

vs $h = \sqrt{\mu a(1-e^2)}$

$$v^2 = GM(\frac{2}{r} - \frac{1}{a})$$

➔ Increases if *r* decreases

Maximum at pericentre

The two-body problem

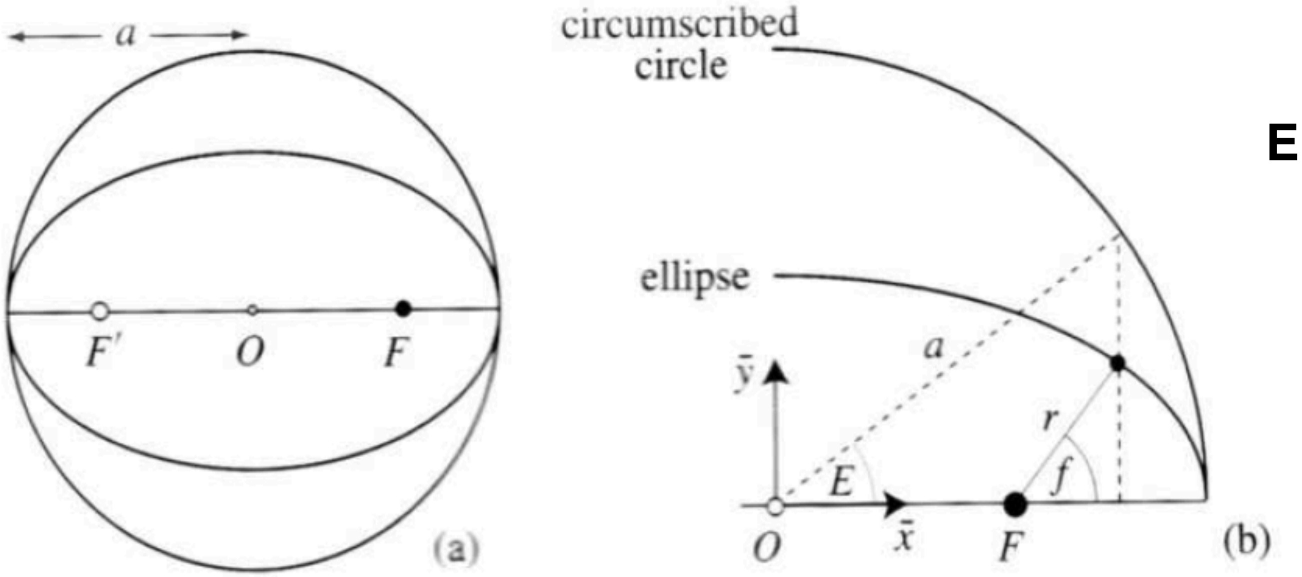
$$\boxed{r = \frac{a(1-e^2)}{1+e\cos f}}$$

Orbital equation does not contain *t*
A relationship between *f* and *t* is thus required

➔ $t = \tau$ Time of pericenter crossing

➔ $M = 2\pi \frac{t-\tau}{P} = n(t-\tau)$

M = mean anomaly
n = mean motion



E = eccentric anomaly

The two-body problem

Equations relating *E*, *f* and *r* :

$$\begin{aligned} r &= a(1 - e\cos E) \\ \cos f &= \frac{\cos E - e}{1 - e\cos E} \\ \tan\frac{f}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \end{aligned}$$

Equation relating M to E :

$$n(t - \tau) = M = E - e\sin E$$

Kepler's equation

Computing the orbital position at a time *t* :

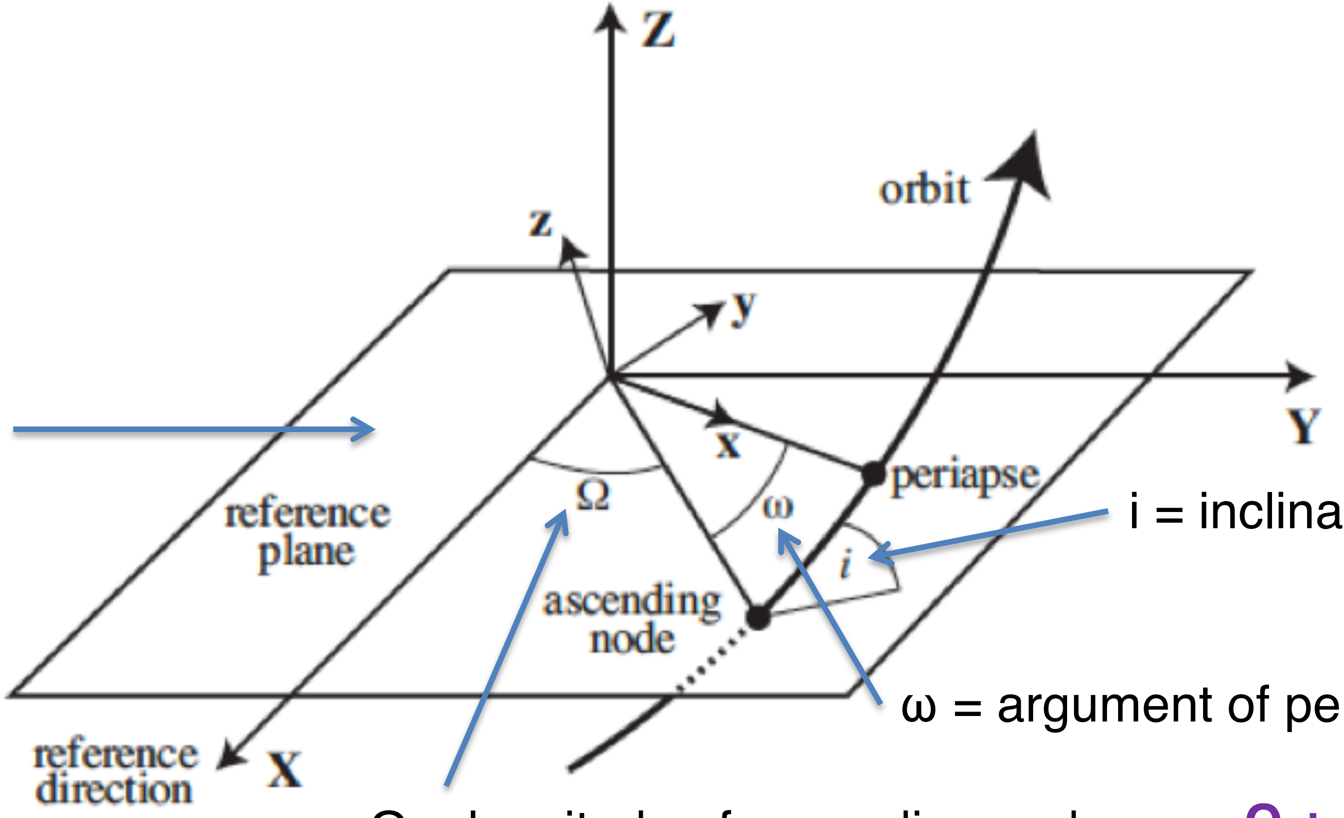
- *a*, *e*, P and the time of pericenter crossing τ are known
- *M* is computed for the time *t*
- Numerical or series (e ~ 0) solution of Kepler's equation ➔ *E*
- Computation of *f* and *r* from *E*

The two-body problem

Motion in 3D

We use 2 cartesian coordinate systems:

$$r = (O_*; \hat{\boldsymbol{x}}; \hat{\boldsymbol{y}}; \hat{\boldsymbol{y}}) \qquad R = (O_*; \hat{\boldsymbol{X}}; \hat{\boldsymbol{Y}}; \hat{\boldsymbol{Z}})$$



Opposite direction to Earth

Plane of the sky

East

North

i = inclination

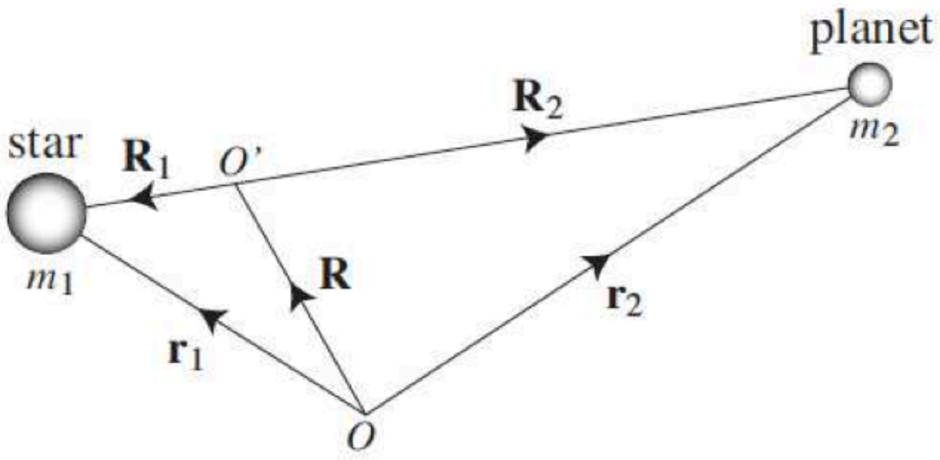
ω = argument of pericentre

Ω = longitude of ascending node

Ω + ω = ϖ

The two-body problem

Motion of the star -> barycentric coordinates



$$\boldsymbol{R_*} = \boldsymbol{r_*} - \boldsymbol{R}$$

$$\boldsymbol{R_p} = \boldsymbol{r_p} - \boldsymbol{R}$$

$$\boldsymbol{R} \equiv \frac{m_* \boldsymbol{r}_* + m_p \boldsymbol{r}_p}{m_* + m_p}$$

$$m_* \boldsymbol{R_*} + m_p \boldsymbol{R_p} = 0$$

Centre of mass lies between the planet and the star

$$\boxed{R_* + R_p = r}$$

$$\boxed{m_* R_* = m_p R_p}$$

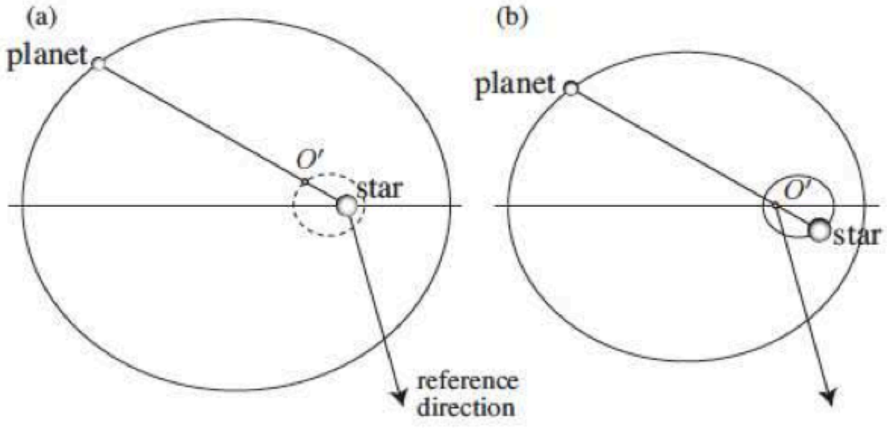
➔

$$R_* = \frac{m_p}{m_p + m_*} r \text{ et } R_p = \frac{m_*}{m_p + m_*} r$$

$$a_* = \frac{m_p}{m_p + m_*} a \text{ et } a_p = \frac{m_*}{m_p + m_*} a$$

The two-body problem

Motion of the star -> barycentric coordinates



$\omega_* = \omega_p + \pi$

Star has an orbit around the CM of the system that is antiphased to the one of the planet..

$$Z_* = \frac{m_p}{m_p + m_*} r \sin(\omega_* + f_*) \sin i$$

$$\boldsymbol{r}_* = \boldsymbol{R} + \boldsymbol{R}_*$$

Radial velocity of the star

$$V_r = \dot{\boldsymbol{r}}_* . \hat{\boldsymbol{Z}} = \gamma_r + \frac{m_p}{m_p + m_*}\left(\dot{r} \sin(\omega + f) \sin i + r \dot{f} \cos(\omega + f) \sin i\right)$$

Systemic velocity

Orbital velocity

The two-body problem

Radial velocity of the star

$$\boxed{V_r = \gamma_r + K\left(\cos(\omega + f) + e\cos\omega\right)}$$

$$K = \frac{m_p \sin i}{m_p + m_*}\frac{na}{\sqrt{1-e^2}}$$

$$K = \frac{m_p \sin i}{m_p + m_*}\frac{a}{\sqrt{1-e^2}}\frac{\sqrt{G}\sqrt{m_p + m_*}}{a^{1.5}}$$

Degeneracy in i

$$\boxed{K = \frac{m_p \sin i}{\sqrt{m_p + m_*}}\sqrt{\frac{G}{a(1-e^2)}}}$$

Varies as $M_*^{-0.5}$

Varies as $a^{-0.5}$

The two-body problem in general relativity

Equation of motion
Newtonian gravitation

$$\frac{d^2u}{d\psi^2} + u = \frac{GM}{h^2}$$

Equation of motion
General relativity

$$\frac{d^2u}{d\psi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

Metric? Schwartzchild: static space-time outside a spherical non-rotating distribution of mass

Mercury has an excess of precession of 43''/century -> very small effect

-> perturbative approach

$$u = \frac{1}{r} = \frac{GM}{h^2}[1 + e\cos\psi] + \Delta u$$

$$\boxed{u \approx \frac{GM}{h^2}\left\{1 + e\cos\left[\psi(1-\alpha)\right]\right\}}$$

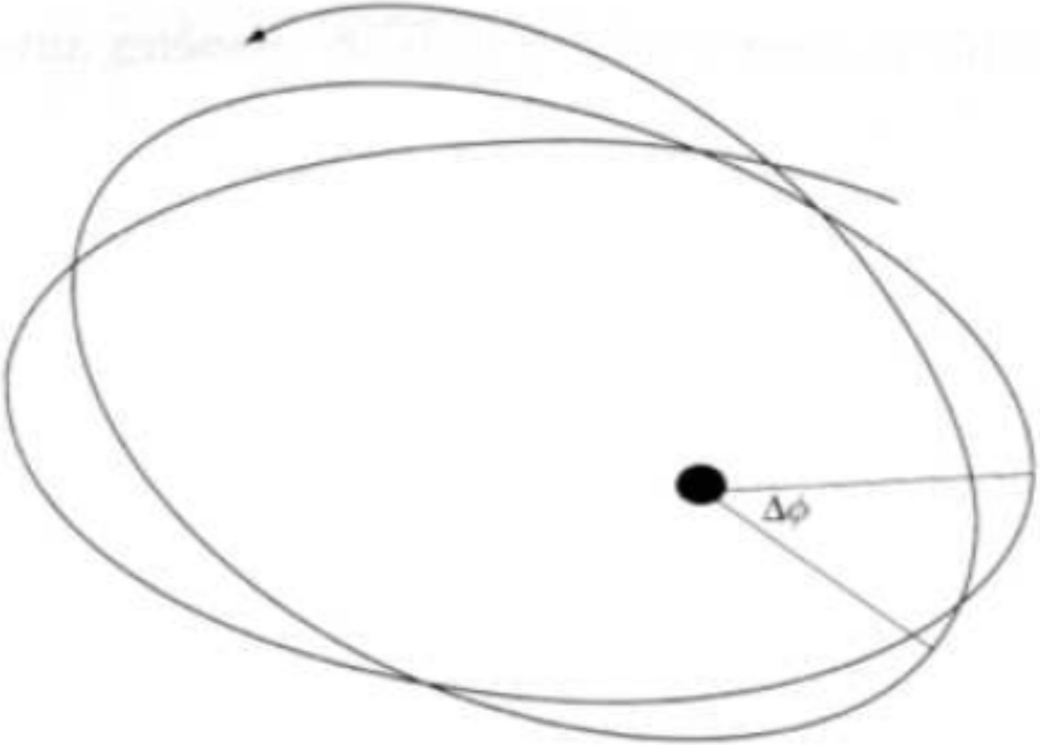
The two-body problem in general relativity

$$u \approx \frac{GM}{h^2}\left\{1 + e\cos\left[\psi(1-\alpha)\right]\right\}$$

The same values come back after a cycle with a phase range larger than 2 π

-> orbit is no more closed

$$\boxed{\alpha = \frac{3(GM)^2}{h^2c^2}}$$



Relativistic precession

The two-body problem in general relativity

Precession from orbit to orbit

$$\delta\psi = 2\pi\alpha = \frac{6\pi(GM)^2}{h^2c^2} = \frac{6\pi GM}{a(1-e^2)c^2}$$

Mercury? *a* = 0.387 UA, *e* = 0.2, M = $1M_\odot$ ➔ 43''/century

Exoplanets ? Some have a very short eccentric orbit

Ex: HAT-P-23b : *a* = 0.0232 UA, *e* = 0.106, M = 1.13 $M_\odot$
➔ **16°/century**

Could be measured within a few dozens years

The three-body problem

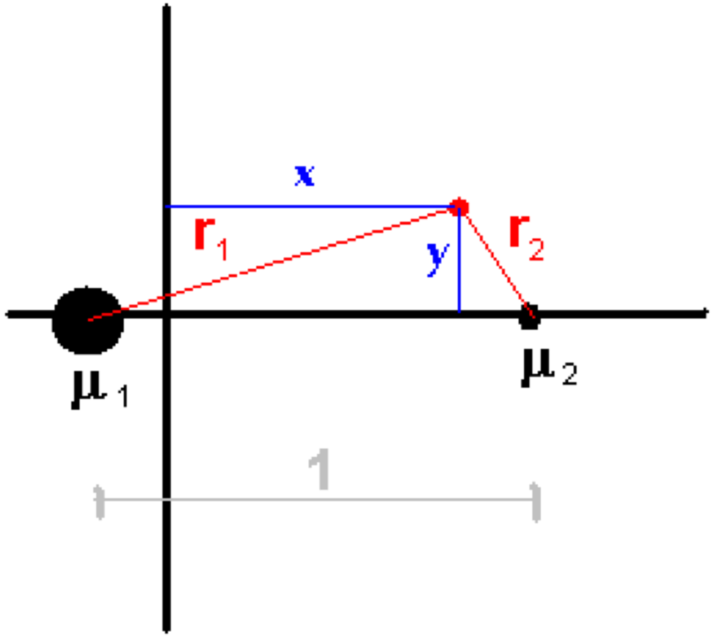
3 bodies ➔ the problem is no more analyticaly tractable

Simplification: 2 bodies in orbit around their common CM + 3rd body = point source

Restricted circular 3-body problem

Allows to tackle the motion of moons, Trojans, ring particules …

Motions are studied within a **synodic** coordinates system **=** centered on the barycenter of M1-M2, in co-rotation with them, and with their distance as unit of distance

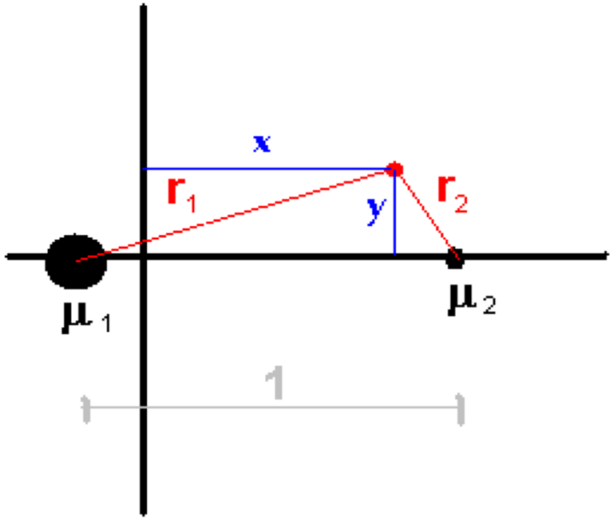


The restricted circular three-body problem

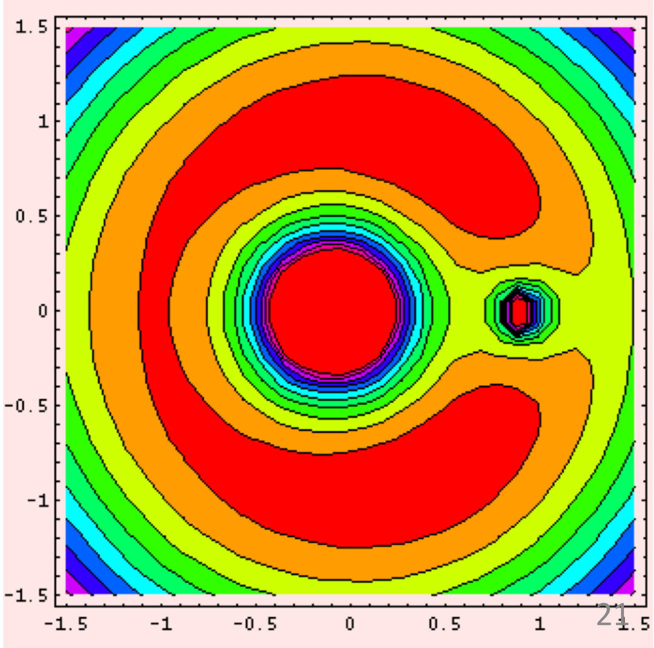
Only 1 constant of the motion = Jacobi constant (or Jacobi integral)

$$C_J = n^2(x^2 + y^2) + 2\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right) - v^2$$

Centrifugal and gravitational potential energy

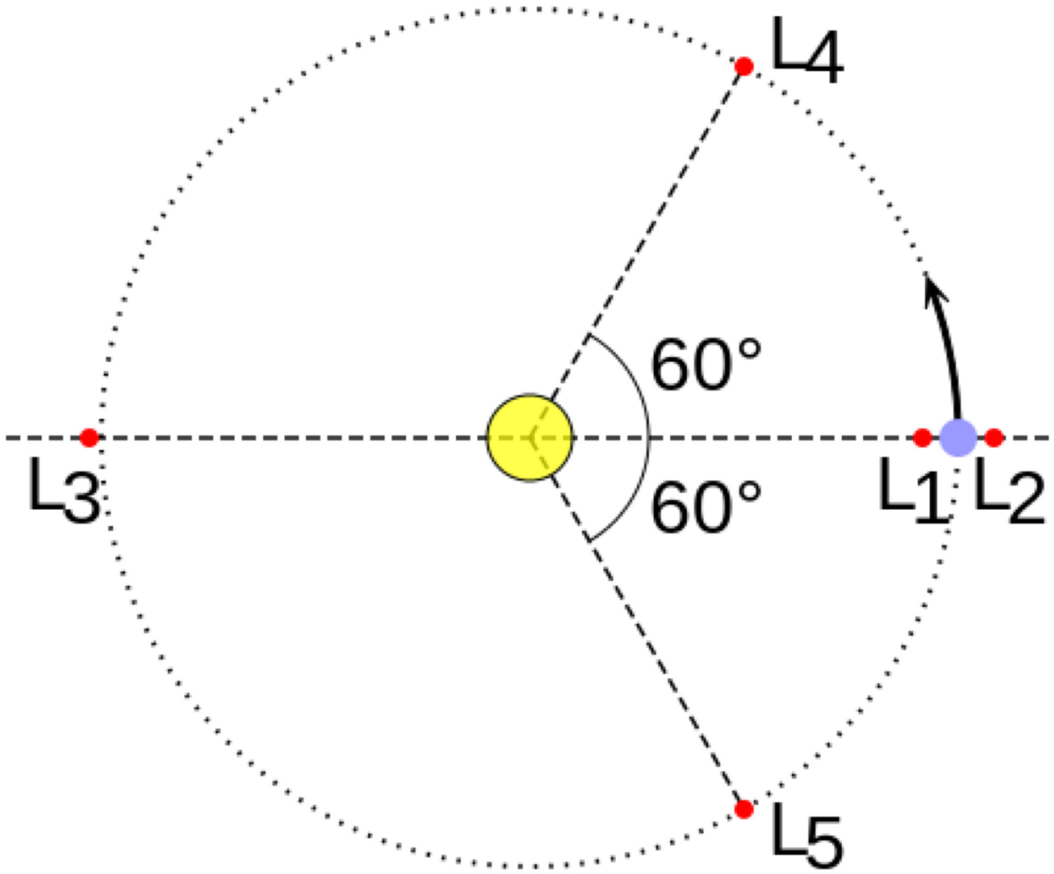


By nulling v^2 for a given C_J are obtained *zero-velocity curves* that delimit the area allowed for the motion of the particule



The restricted circular three-body problem

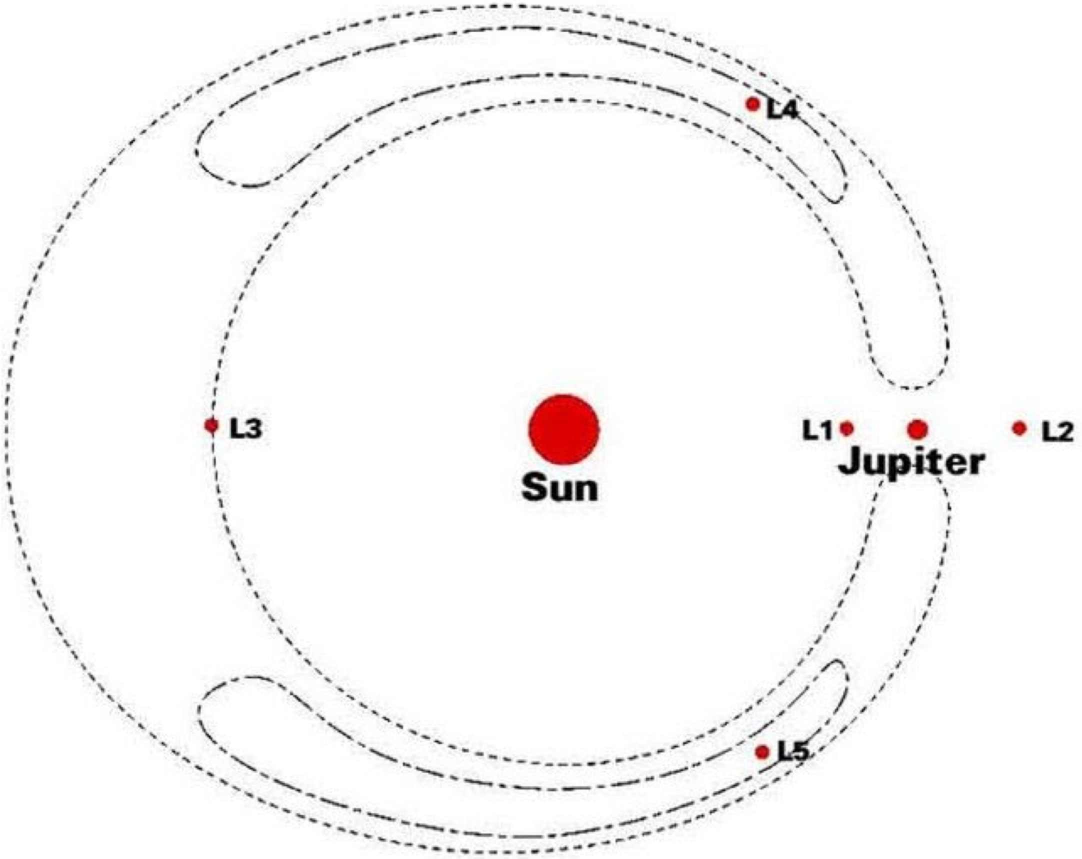
5 equilibrium points = Lagrangian points



The points L_1, L_2 et L_3 are unstables. **L_4 et L_5 are stables for $m_1/m_2 \geq 27$**

The restricted circular three-body problem

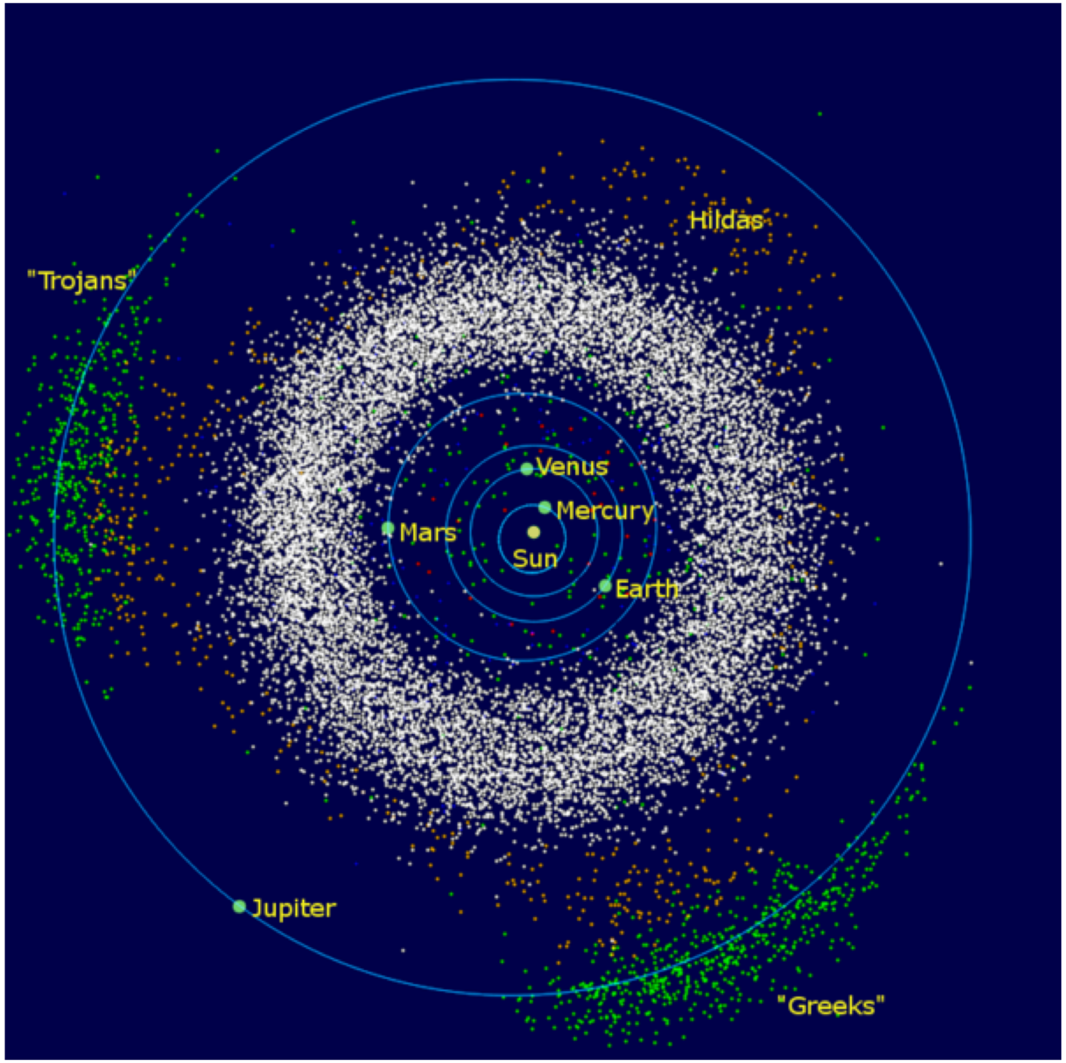
Trojans: Libration around the points L4 et L5



« Tadpole » and « horseshoe » orbits

The restricted circular three-body problem

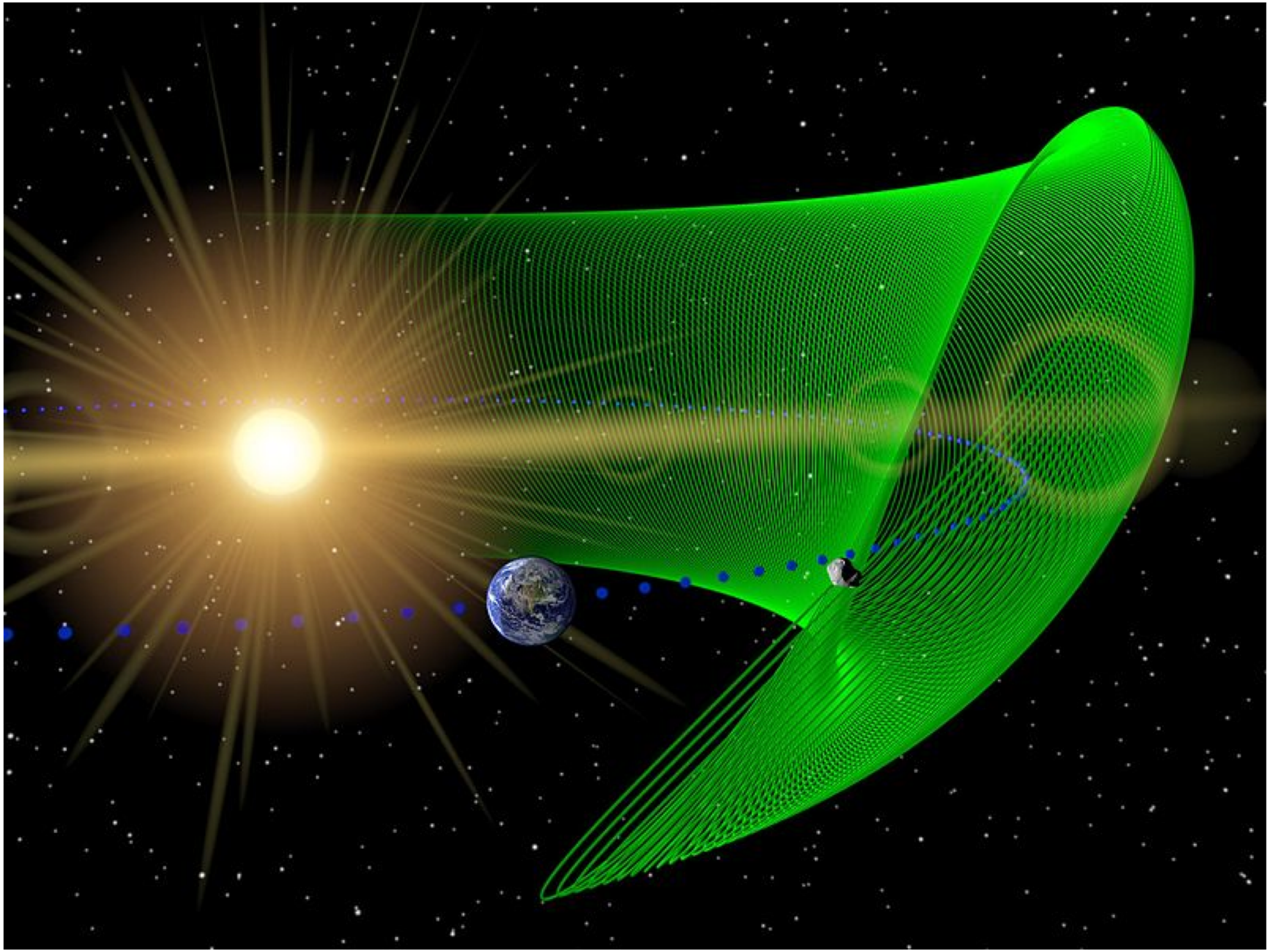
Tadpole orbit: Jupiter's Trojans (more than 2000!)



Also known for Uranus, Neptune, Mars, and the Earth

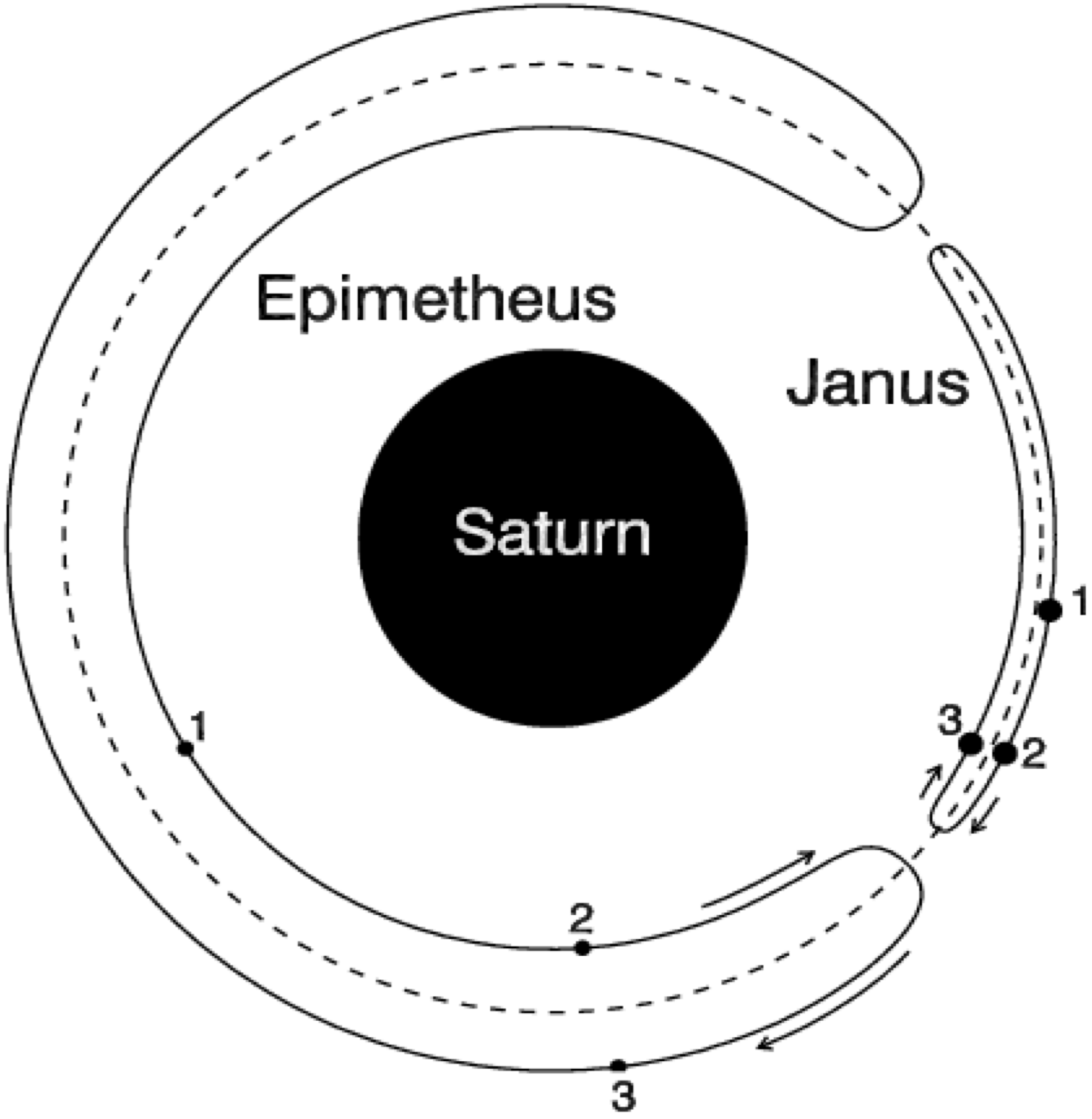
Earth's Trojan

2010 TK$_7$: a 300m-size asteroid librating around the Earth's L$_4$ point!



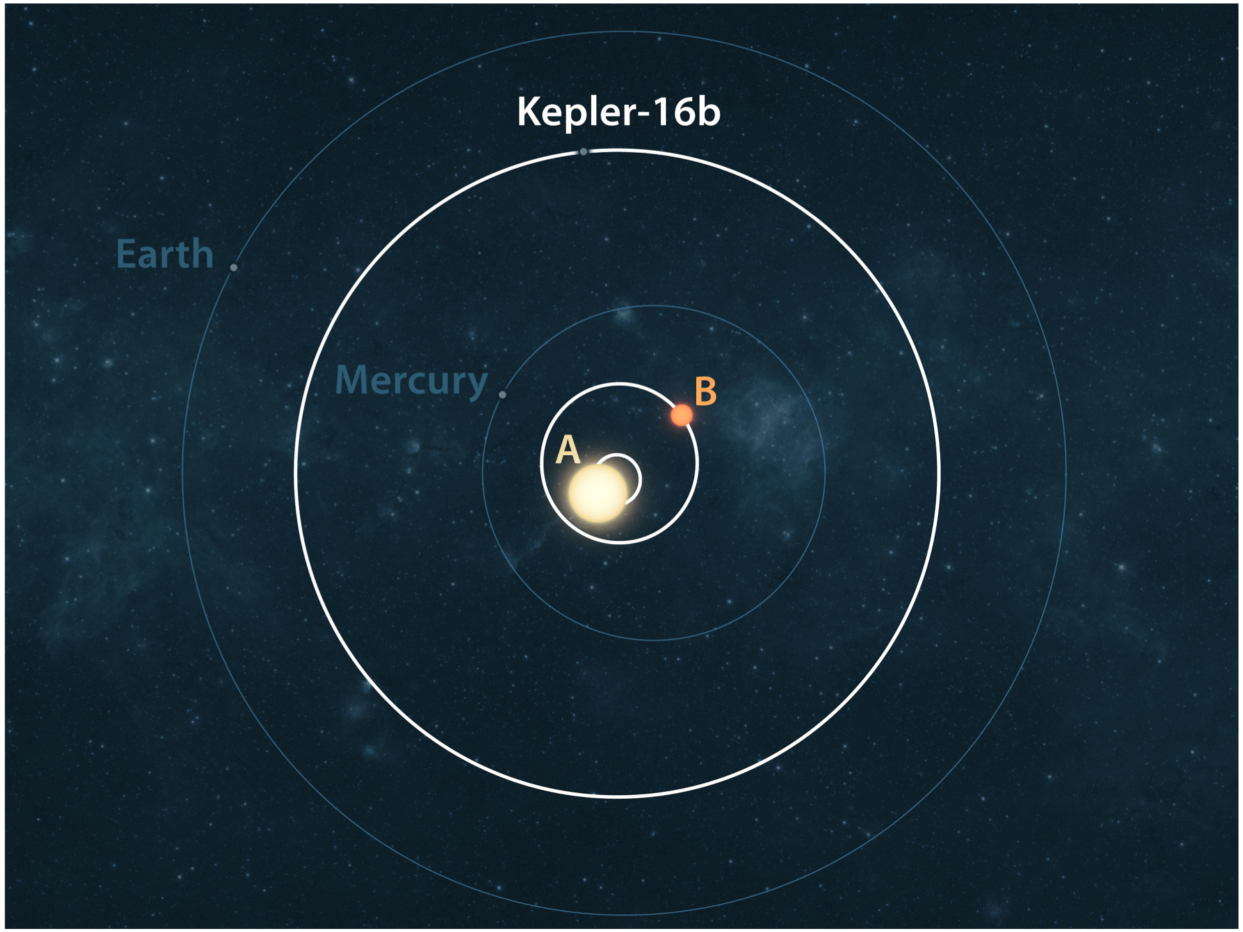
The restricted circular three-body problem

Horseshoe orbits: the Janus-Epimetheus example



The restricted circular three-body problem

Circumbinary orbits: about 30 known so far



Kepler-16A and B :
1 K-type and 1 M-type star in a 41d circular orbit

Kepler-16(AB)b:
A Saturn-mass planet in a 229d orbit around the binary

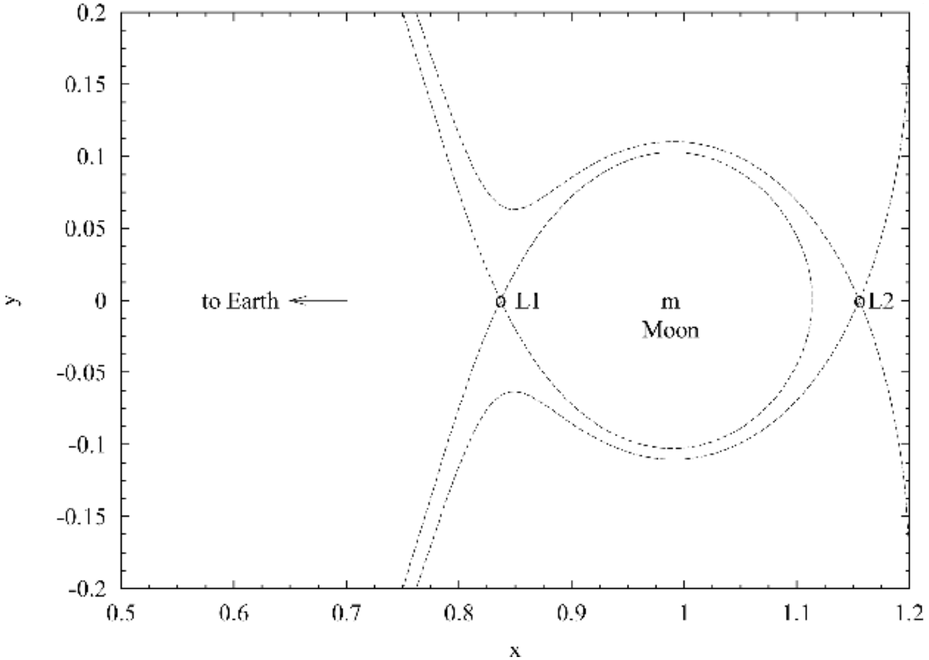
Other examples: Kepler-35, 38, 47, …

The Hill radius R_H

Limit distance beyond which the particule can no more remain in orbit around m_2. It corresponds to the distance m_2-L_1

$$R_H = \left(\frac{m_2}{3(m_1+m_2)}\right)^{1/3} a$$

Practically, a planetocentric orbit is stable if $R << R_H$. The maximum distance for a stable orbit is larger is the orbit is *retrograde*.



The N-body problem

No analytical solution ➔ numerical integration of the equations of motion is the general approach

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1,j\neq i}^{j=N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}.$$

Practically, **symplectic integrators** are often used, i.e. algorithms integrating at each step the Hamilton equations while ensuring the conversation of key quantities like energy.

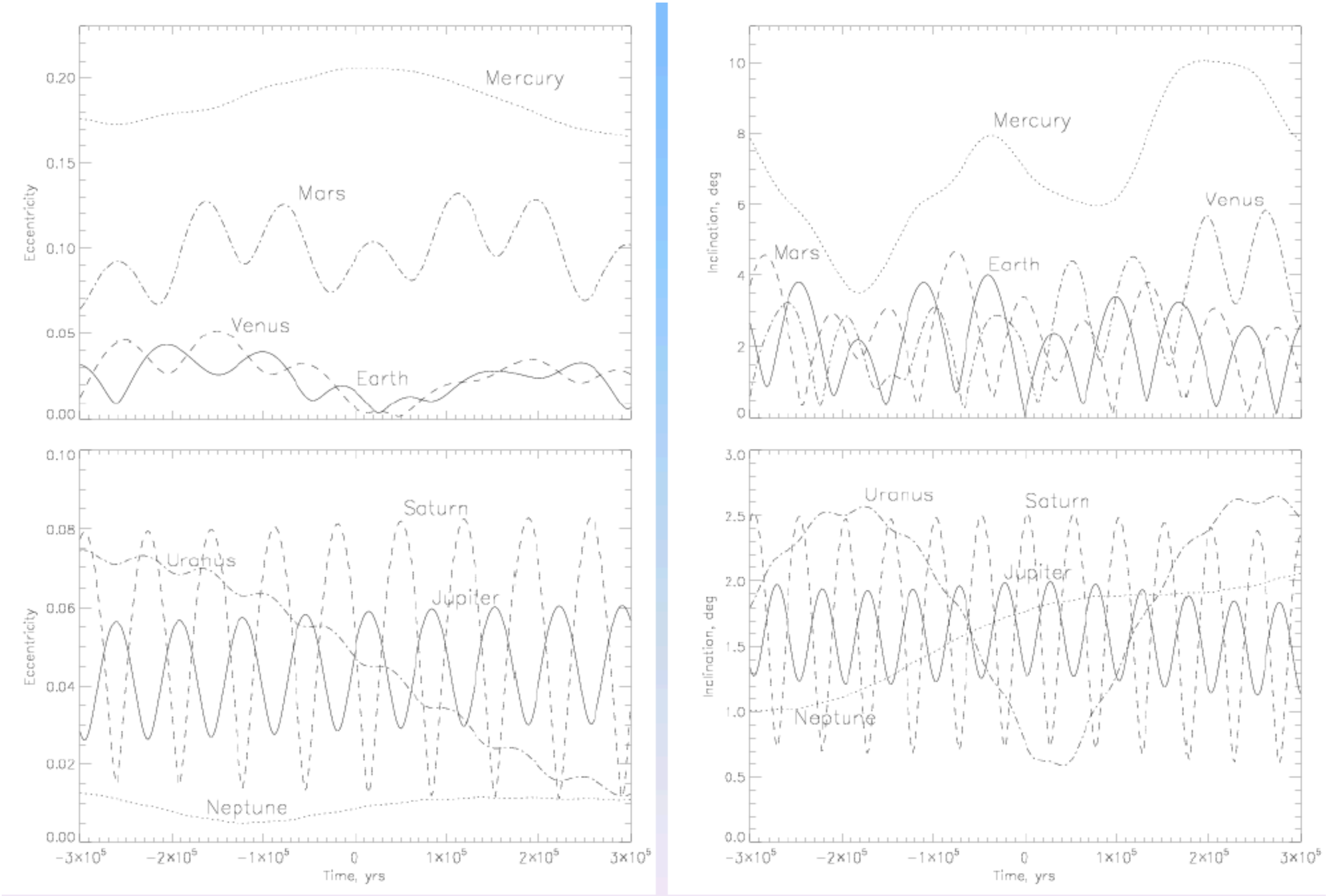
$$\boxed{\dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p}}$$

H = Hamiltonian, which corresponds to total energy of the system.

p and *q* are canonical coordinates

Secular evolution

Assumption: interactions within orbits can be averaged and we study the evolution of the averaged orbits = **secular evolution**



Correlated variations of *e* and *i*. **Exchange of angular momentum**

Resonances

Regular, periodic, gravitational influence between 2 or more bodies due to some of their orbital parameters being related by an integer ratio

Ex: orbital resonances (Galilean moons)
spin-orbit resonance (Moon)

Orbits do not average anymore, each orbit matters

Analogy: forced harmonic oscillator

$$m\frac{d^2x}{dt^2} + m\omega_o^2 x = F_f \cos\omega_f t$$

Si $\omega_f \neq \omega_o$

$$x = \frac{F_f}{m(\omega_o^2 - \omega_f^2)}\cos\omega_f t + C_1 \cos\omega_o t + C_2 \sin\omega_o t$$

Si $\omega_f = \omega_o$

$$x = \frac{F_f}{2m\omega_o}\boxed{t}\cos\omega_o t + C_1 \cos\omega_o t + C_2 \sin\omega_o t$$

Cumulative effects do not only make possible exchange of angular momentum but also of orbital energy

Orbital resonances

Consider two planets in circular coplanar orbits with

$$\boxed{\frac{n_2}{n_1} \approx \frac{p}{p+q}}$$

with n_i=$2\pi/P_i$ is the mean motion, and p and q are two integers.

If conjunction at $t = 0$, next conjunction when n_1t - $n_2t = 2\pi$
So the time difference betwen 2 conjunctions is

$$\Delta T = \frac{2\pi}{n_1 - n_2} = \frac{2\pi}{n_1 \frac{q}{p+q}} = \frac{p+q}{q} P_1$$

And thus

$$q\Delta T = (p+q)P_1 = pP_2$$

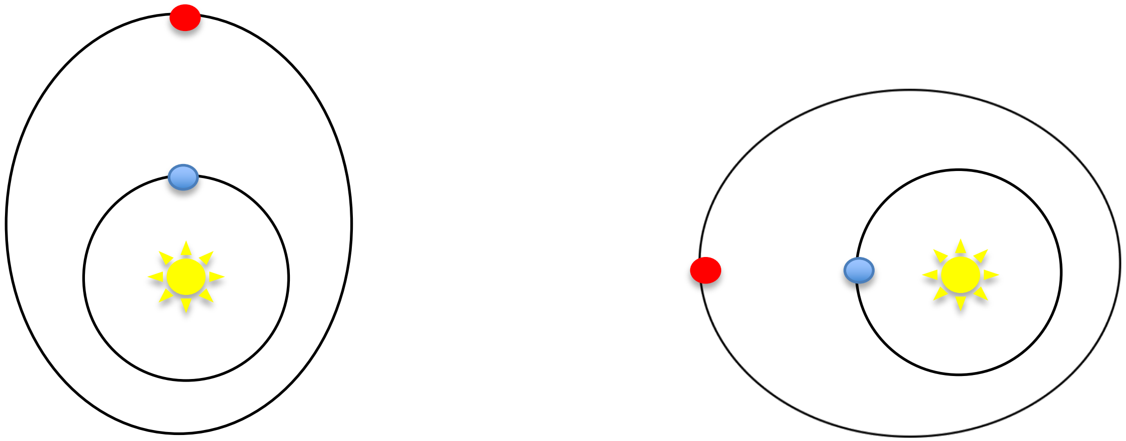
Each *q*-th conjunction occurs at the same longitude.
q = resonance order

Orbital resonances

If the outer planet has $e_2 \neq 0$ and $\dot{\varpi}_2 \neq 0$, resonance if

$$\frac{n_2 - \dot{\varpi}_2}{n_1 - \dot{\varpi}_2} = \frac{p}{p+q}$$

In this case, we have: $(p+q)n_2 - pn_1 - q\dot{\varpi}_2 = 0$



Each *q*-th conjonction takes place at the same true anomaly for the outer planet, but it does not correspond anymore to the same longitude, i.e. to the same point in an inertial system.

The commensurability of orbital periods does not automatically mean true orbital resonance (precession).

Orbital resonances

Effect of resonances: stabilization

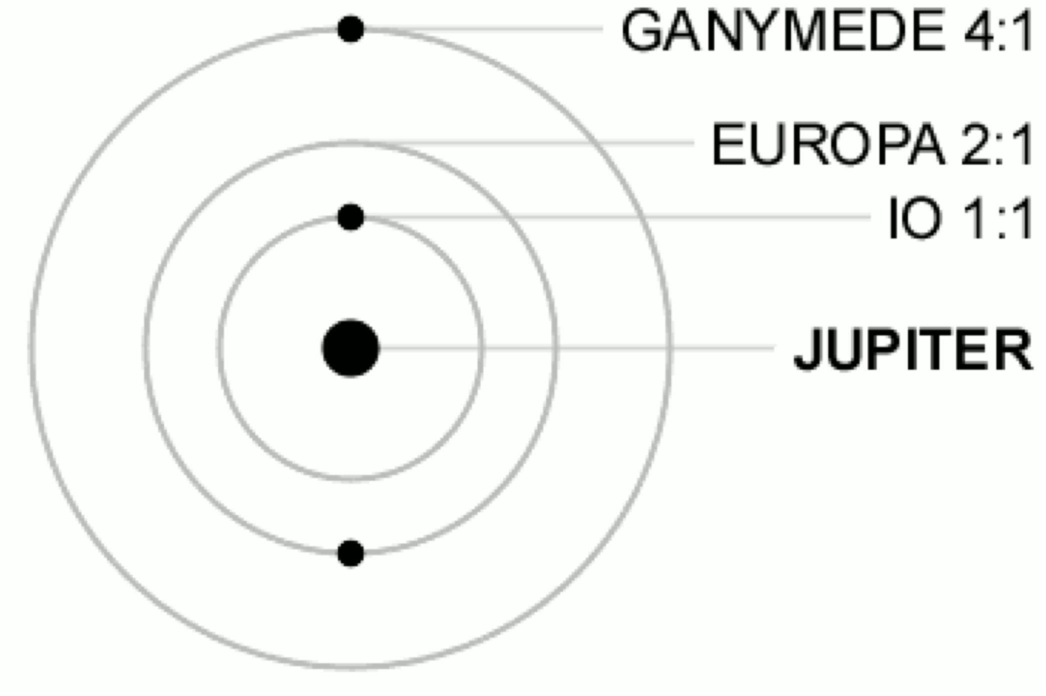
ex: Jupiter-Io-Europa-Ganymede

$$\lambda_I - 2\lambda_E + \varpi_I = 0°, \quad \lambda_I - 2\lambda_E + \varpi_E = 180°, \quad \lambda_E - 2\lambda_G + \varpi_E = 0°,$$

$$n_I - 2n_E + \dot{\varpi}_I = 0, \quad n_I - 2n_E + \dot{\varpi}_E = 0, \quad n_E - 2n_G + \dot{\varpi}_E = 0$$

Laplace's relationships:

$$\boxed{\begin{gathered}\phi_L = \lambda_I - 3\lambda_E + 2\lambda_G = 180°, \\ n_I - 3n_E + 2n_G = 0\end{gathered}}$$



Ever triple conjunction

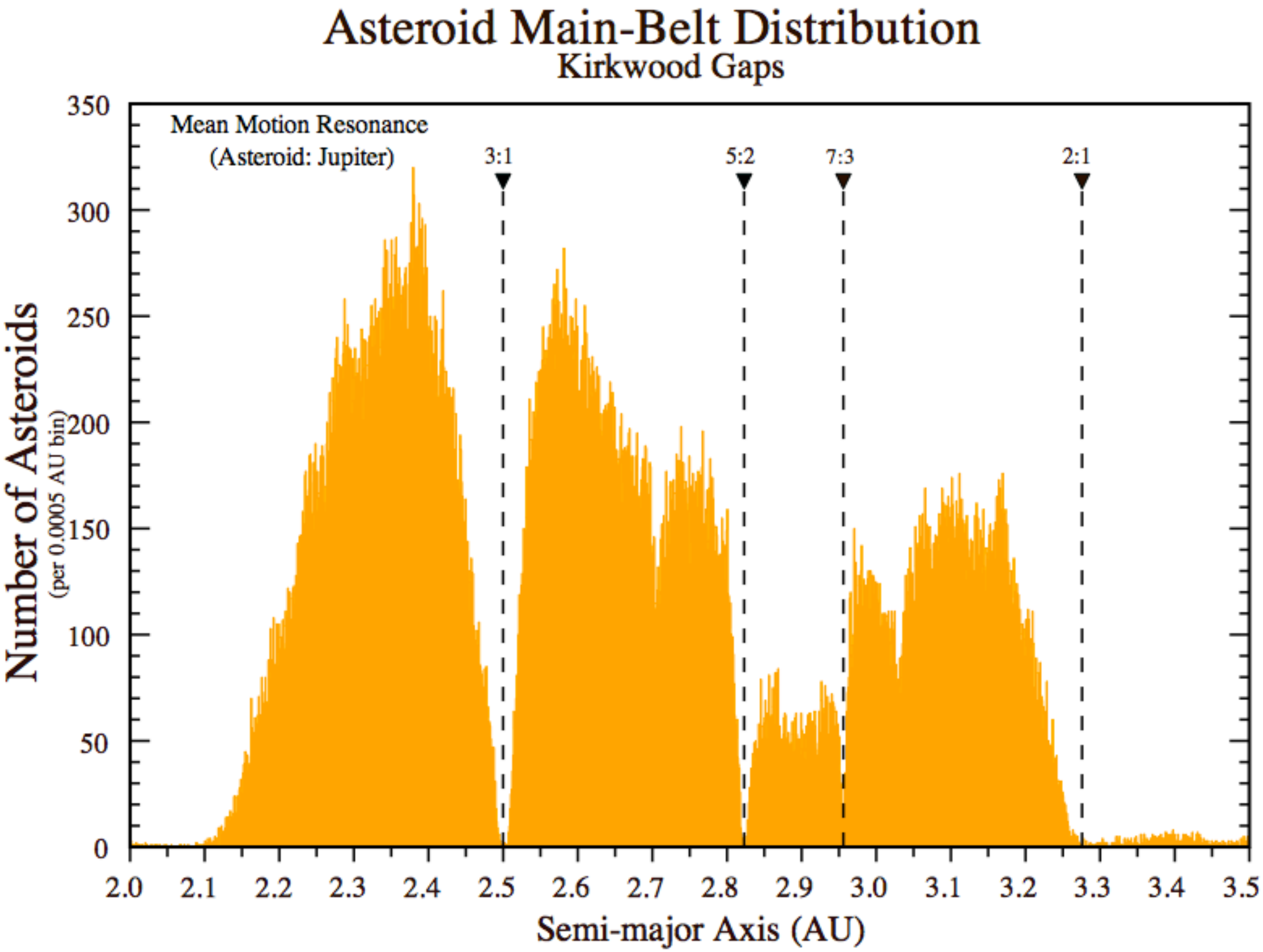
Libration of ϕ$_L$ with a period of 2017 days and with an amplitude of 0.064°

Maintains the eccentricity of Io (0.004) and Europa (0.01)

Orbital resonances

Effect of resonances : destabilization

ex: Kirkwood's gaps

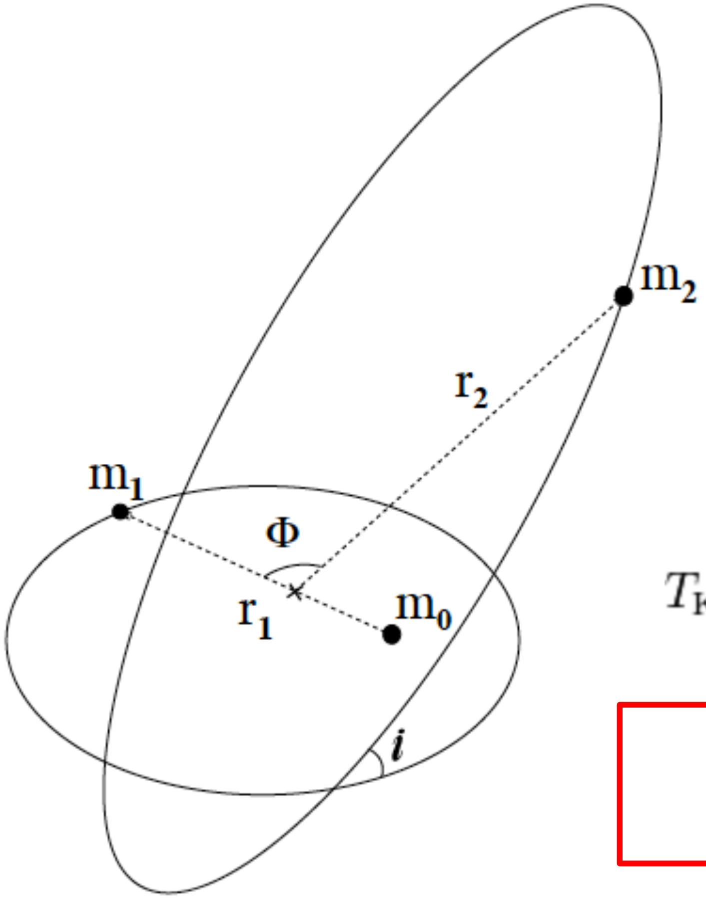


Chaotic orbits

Kozai mechanism

Star with planet, + a star or a massive planet on a outer and very inclined orbit (>39°)

Coupled oscillation of *e* and *i* of the inner planet



Ford et al. (2000)

Oscillations of *e* and *i* with

$$L_z = \sqrt{(1-e^2)}\cos i$$ conserved

$$T_{\rm Kozai} = 2\pi \frac{\sqrt{GM}}{Gm_2}\frac{a_2^3}{a^{3/2}}\left(1-e_2^2\right)^{3/2} = \frac{M}{m_2}\frac{P_2^2}{P}\left(1-e_2^2\right)^{3/2}$$

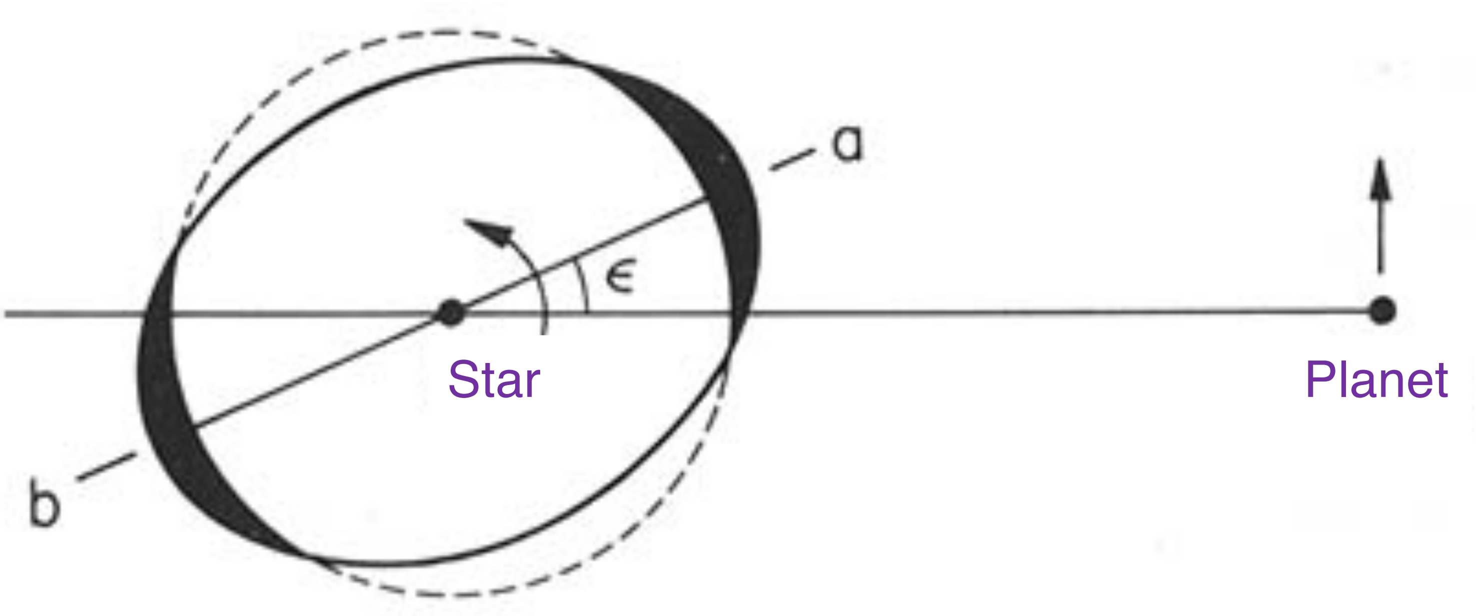
Mechanism able to produce eccentric Jupiters and hot Jupiters

Tidal effects

Are assumed a star and a close-in planet.

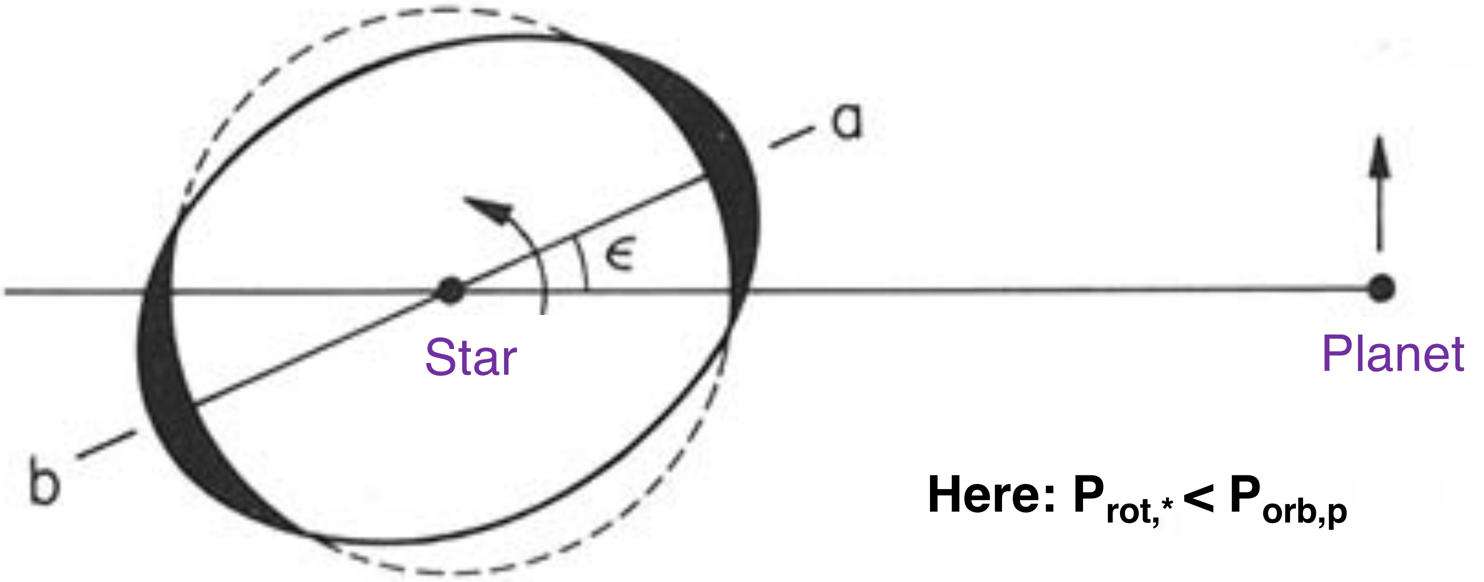
The star distorts the planet, and reciprocally ➔ tidal bulges

The two bodies have a non-zero viscosity ➔ friction forces
➔ heating and phase shift of the bulges



Here : $P_{rot,*} < P_{orb,p}$

Tidal effects



Here: $P_{rot,*} < P_{orb,p}$

with $2\varepsilon = Q^{-1}$

where Q = **tidal dissipation function**

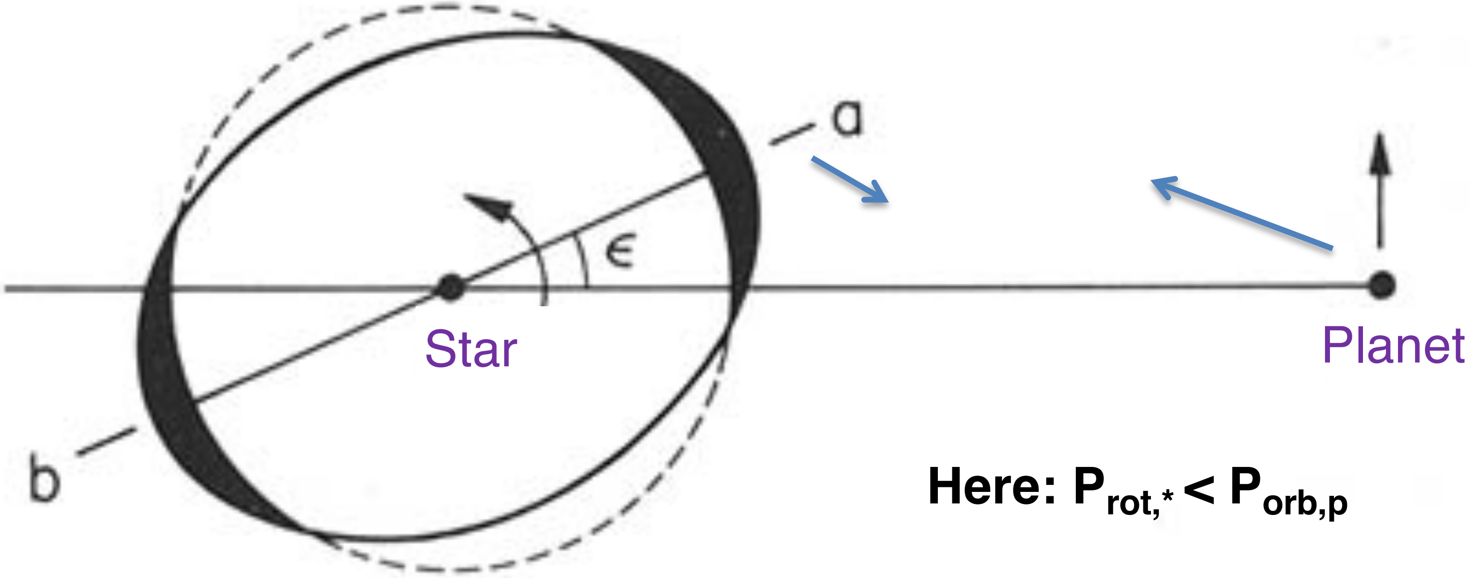
= maximum energy stored in the tidal deformation over the tidal energy dissipated as heat per cycle

= 10 – 500 for terrestrial bodies

> 10^5 for giant planets and stars (much more fluid)

Note: Q depends on the orbital period too

Tidal effects



Here: $P_{rot,*} < P_{orb,p}$

The tidal deformation of the star results in a torque that accelerates the planet and slows down the stellar rotation (in the case of the Earth-Moon system)

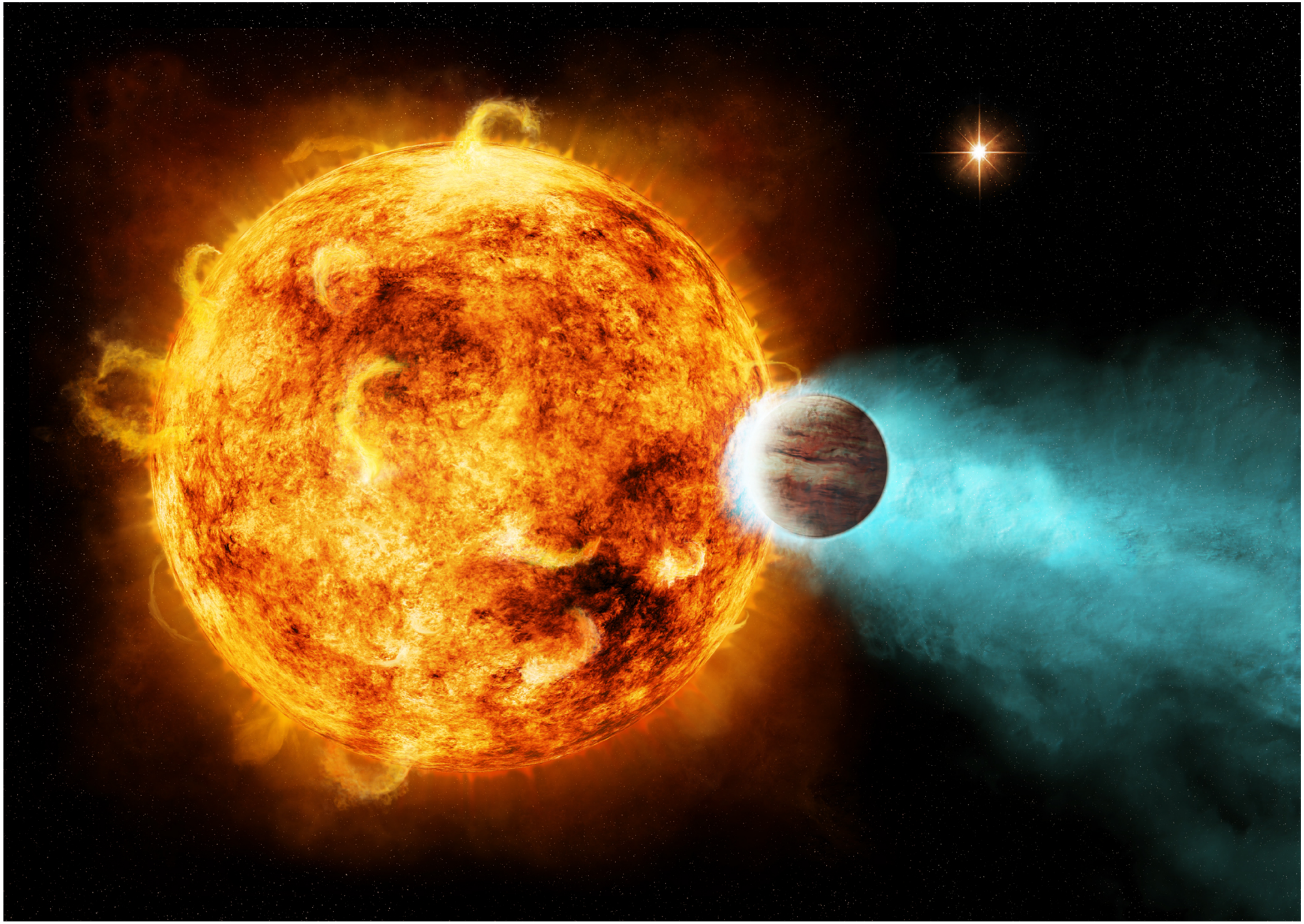
➔ Transfert of energy and angular momentum between the two bodies

➔ Here $P_{rot,*}$ and $P_{orb,p}$ increase, in the opposite case they decrease

➔ Variation of $P_{rot,*}$, $P_{rot,p}$, I_*, I_p, a, e

➔ Final outcome: **complete equilibrium ($P_{rot,*} = P_{rot,p} = P_{orb}$; $I_* = I_p$; $e = 0$)** or **tidal disruption** (hot Jupiters) or **damped orbital recession** (Moon)

Tidal evolution of hot Jupiters ($P_{orb} < P_{rot,*}$)



Tidal evolution of hot Jupiters ($P_{orb} < P_{rot,*}$)

1. Very fast evolution towards $P_{orb} = P_{rot,p}$ in ~1Ma
➔ **spin-orbit resonance (tidal locking)**

2. **Much slower circularization of the orbit** within a timescale of ~1Ga

3. **Continuous shrinking of the orbit** due to tides raised by the planet on the star (making the star rotate faster)

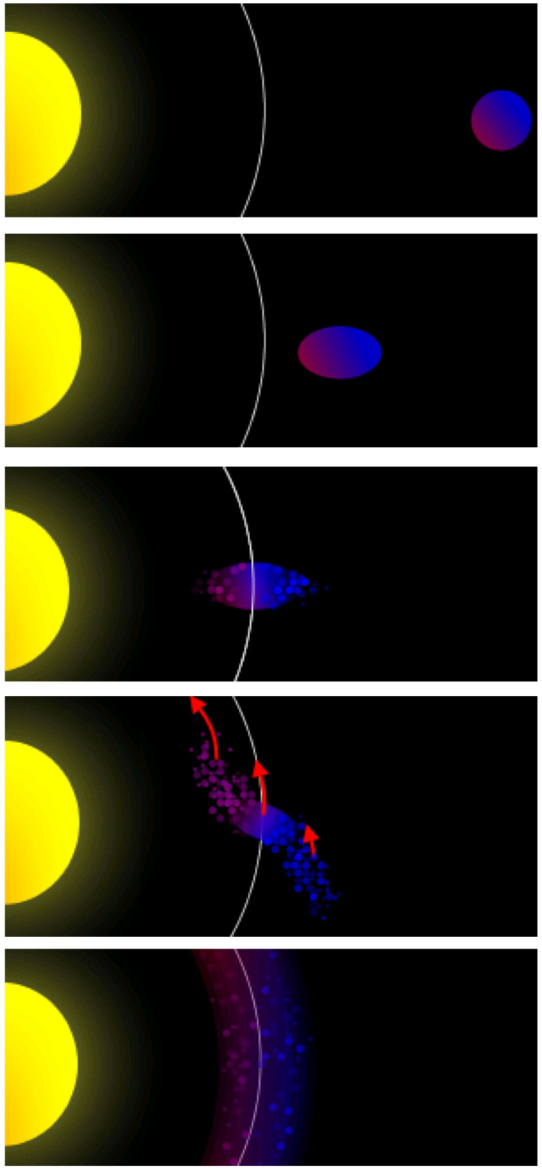
4. $P_{rot,*}$ is modified by tidal effects (acceleration), but also by stellar wind (magnetic braking), so **complete equilibrium is never reached** and $da/dt < 0$

Final outcome: tidal disruption

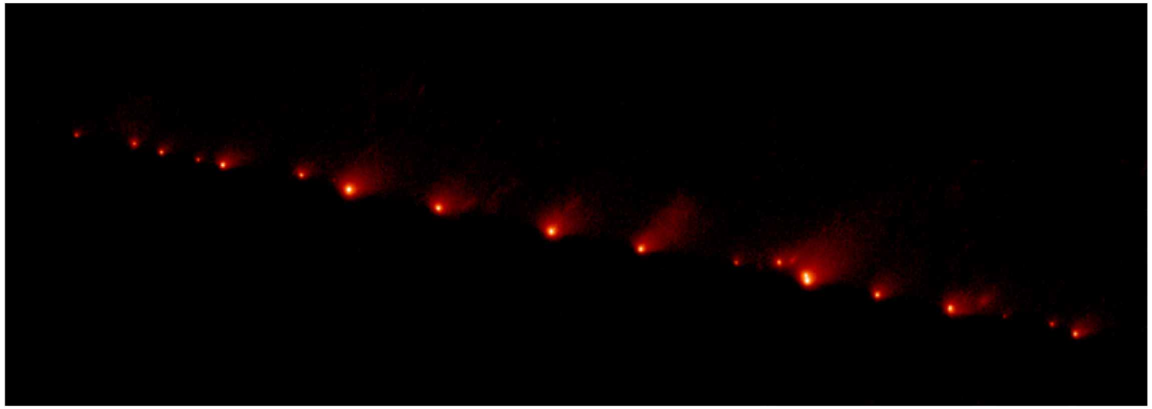
Rocky planets? Evolution is much slower because of much smaller tides on the star + much less energy dissipated per cycle (e.g. Mercury with e=0.21)

Tidal disruption

The planet migrates until reaching its **Roche limit,** distance for which the stellar gravity and the centrifugal forces surpass its internal cohesion forces



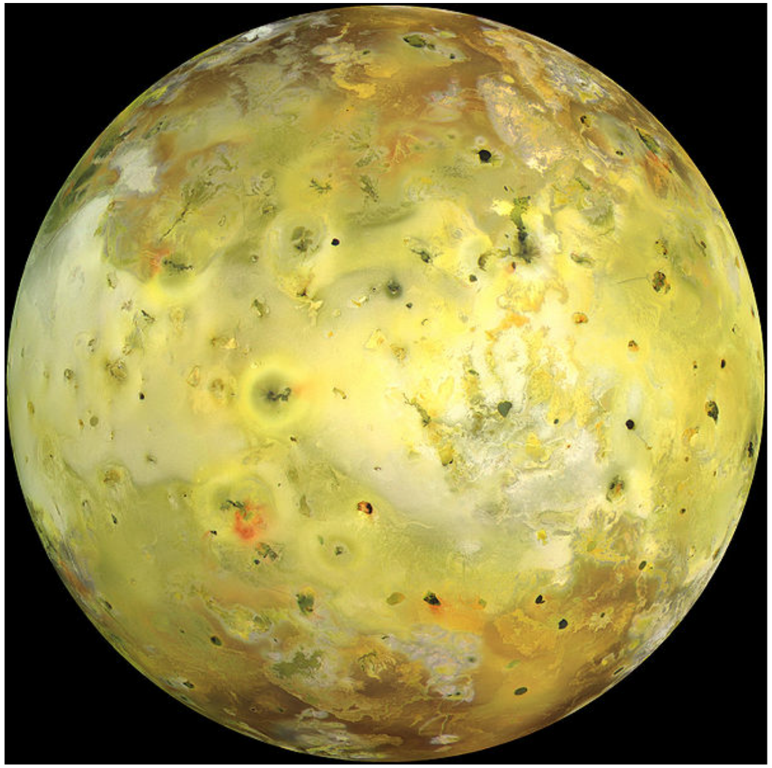
$$d \approx 2.44 R_M \left(\frac{\rho_M}{\rho_m} \right)^{1/3}$$



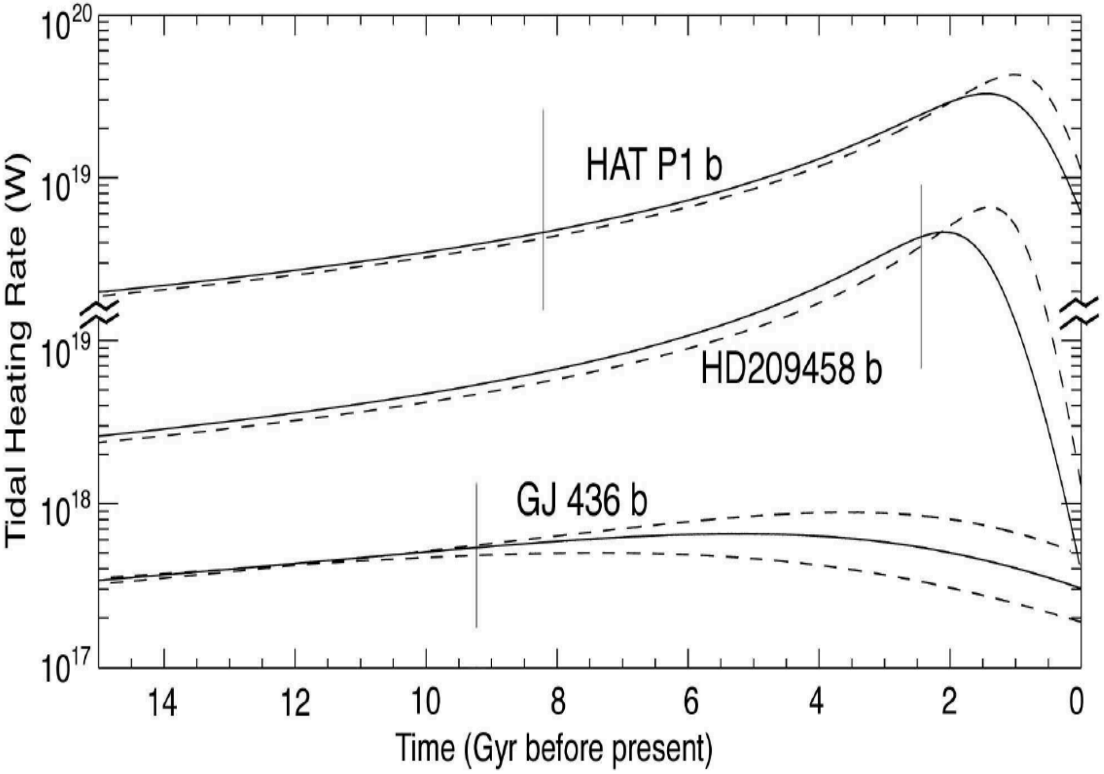
Shoemaker-Levy 9 comet (17/05/1994)

If differentiated planet: only the outer layers are torn apart ➔ *chtonian planet*

Tidal heating



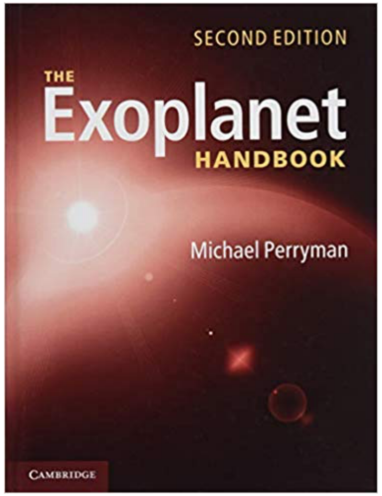
$$H = \frac{63}{4}\frac{(GM_*)^{3/2}M_*R_p^5}{Q_p^{'}}a^{-15/2}e^2$$



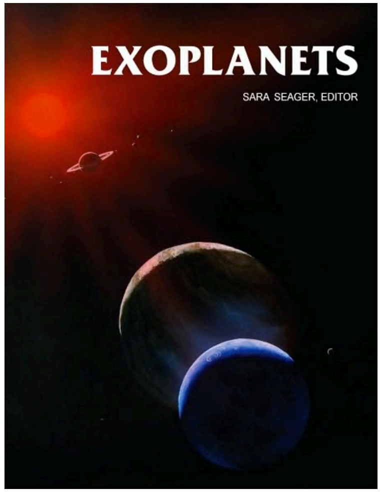
Jackson et al. (2009b)

Important for energy budget of short-period planets

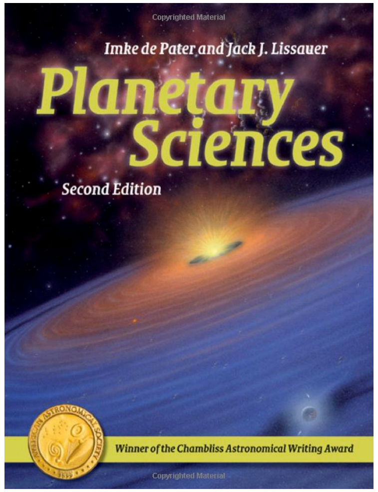
References



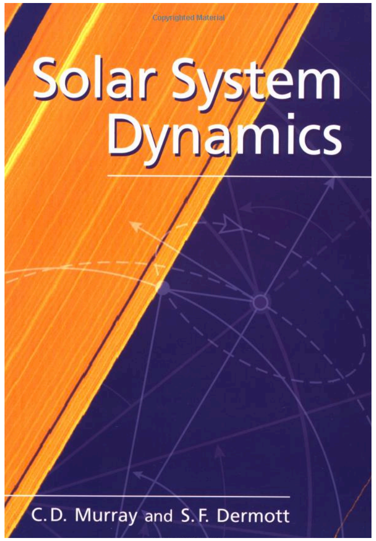
M. Perryman
Cambridge University Press
Chapters 2, 3, 6 & 10



S. Seager
University of Arizona Press
Chapters 2, 10 & 11



I. de Pater & J. J. Lissauer
Cambridge University Press
Chapter 2



C. D. Murray & S. F. Dermott
Cambridge University Press