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Influence of Convection on the Vibrational Stability of Stars towards Non-radial Oscillations

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Summary. The coupling between convection and non radial oscillations is discussed. Limitations of the theory come from the lack of a good theory for turbulent convection.

Key words: stellar stability

I. Introduction

When the vibrational stability of stars towards radial or non radial oscillations is studied, the difficult problem of the coupling between convection and pulsation is nearly always encountered. For radial pulsations the problem has been considered by Boury *et al.* (1964), Cox *et al.* (1966) and Unno (1967). However due to our poor knowledge of convection even for non pulsating stars it is impossible at the present time to solve the problem in a satisfactory way. The limitations are mainly due to the use of the Boussinesq approximation and of the Böhm-Vitense theory for the unperturbed convection.

We will follow here an approach similar to Unno's, which is the best possible one at the moment and we will extend his work to non radial oscillations.

First we deduce the equations for the pulsation and for the convection using essentially the same approximation as Unno but keeping the equations in a form slightly more general because they have to be solved for a non radial motion (Section II). It is then briefly recalled (Section III) that the equations for convection admit a stationary solution equivalent to the Böhm-Vitense theory if they are solved in a local approximation. In Section IV the equations are perturbed and solved, in a local approximation. In Section VI the results are compared to these obtained in previous works and finally the importance of convective terms in σ' is discussed (Section VII).

II. Fundamental Equations

To deduce the equations of the problem we shall follow the same procedures as Unno (1967). First we write the basic equations of hydrodynamics with

$$\varrho=\bar{\varrho}+\Delta\varrho \qquad p=\bar{p}+\Delta p \qquad T=\bar{T}+\Delta T \qquad \boldsymbol{v}=\boldsymbol{u}+\boldsymbol{V}\,. \tag{1}$$

$\bar{\varrho}$, $\bar{T}$, $\bar{p}$ and $\boldsymbol{u}$ are average values taken over a surface whose sizes are much larger than the characteristic lengths of convection but much smaller than those of the perturbation. Since the problem is spherically symmetric in an unperturbed star, this surface will be chosen spherical. When a perturbation is applied, the average values are defined on the surface resulting from the deformation of the spherical surface considered in the unperturbed state through the perturbation. $\Delta\varrho$, ΔT, Δp and $\boldsymbol{V}$ are the local fluctuations due to convection.

Then, we average these equations to obtain the relations for mean quantities, first deduced by Cowling (1939).

Finally, the subtraction of these equations from the corresponding general ones gives the equations for the fluctuations. Several approximations are however made in this step: the Boussinesq approximation, the neglect of Δp except in the equation of motion and the assumption that the turbulent velocity is largely subsonic. Moreover, we assume with Unno (1967) that

$$\sum_i\left(\varrho V^i \nabla_i V^j-\frac{\varrho}{\bar{\varrho}}\,\overline{\varrho V^i \nabla_i V^j}\right)=\frac{\alpha}{\tau}\,\varrho V^j\,, \tag{2}$$

$$\sum_i\left(\varrho V^i \nabla_i \Delta S-\frac{\varrho}{\bar{\varrho}}\,\overline{\varrho V^i \nabla_i \Delta S}\right)=\varrho\,\frac{\Delta S}{\tau}\,, \tag{3}$$

$$\varepsilon-\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}_R-\frac{\varrho}{\bar{\varrho}}\,\overline{\left(\varepsilon-\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}_R\right)}=-\omega_R T\Delta S\,, \tag{4}$$

where α is a number of the order of unity, τ is the mean life time of turbulent eddies.

Finally using the notation

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla \tag{5}$$

we obtain the equation for the fluctuations

$$\nabla \cdot V = 0, \tag{6}$$

$$\varrho \frac{dV^j}{dt} = \sum_i g^{ji} \left(\frac{\Delta \varrho}{\varrho} \nabla_i \bar{p} - \nabla_i \Delta p \right) - \frac{\alpha}{\tau} \varrho V^j - \sum_i \varrho V^i \nabla_i u^j, \tag{7}$$

$$\frac{\Delta(\varrho T)}{\bar{\varrho}\bar{T}} \frac{d\bar{S}}{dt} + \frac{d\Delta S}{dt} + V \cdot \nabla \bar{S} = -\frac{(\omega_R \tau + 1)}{\tau} \Delta S, \tag{8}$$

$$(\nabla - \nabla_R) \nabla \ln p = \frac{1}{2} \omega_R^{-1} \mathscr{L}^{-2} \overline{\frac{\Delta T}{T} V}, \tag{9}$$

$$\Delta S = C_p \frac{\Delta T}{\bar{T}}, \tag{10}$$

$$\frac{\Delta T}{\bar{T}} = Q \frac{\Delta \varrho}{\bar{\varrho}}, \tag{11}$$

with

$$Q = \left(\frac{\partial \ln T}{\partial \ln \varrho} \right)_p, \tag{12}$$

$$\nabla = \frac{d \ln T}{d \ln p}, \tag{13}$$

$$\omega_R = \frac{4ac}{3} \frac{\bar{T}^3}{C_p \bar{\varkappa} \bar{\varrho}^2 \mathscr{L}^2}. \tag{14}$$

∇_R is the gradient ∇ which would be required if the radiation were to transport all the energy and $\mathscr{L}$ is a characteristic length of the eddies.

Equations (6)–(11) are similar to these deduced by Unno (1967). They however differ on a few points: First, Unno writes them in a form appropriate for applications to radial oscillations while here we must keep them in a more general form. Secondly the last term of the right hand side of Eq. (7) is neglected by Unno. The results of this difference will be discussed in Section VII. Finally to deduce Eq. (8) we do not neglect ΔC_p and $\Delta \left(\frac{\partial T}{\partial p} \right)_S$, therefore our expression for energy conservation is more widely valid than Unno's.

III. Solution for Stationary Convection

Unno (1967) had already shown that, when Eqs. (6)–(11) are solved in the local approximation, they have a stationary solution identical to the Böhm-Vitense theory, owing essentially to the assumptions made in Eqs. (2)–(4).

If f is any of the variables $\Delta \varrho, \Delta T, \Delta p, \Delta S, V^i$, the solution is of the form

$$f(r) = f_a \exp[i \boldsymbol{k} \cdot \boldsymbol{r}], \tag{15}$$

where f_a is a constant

The relations useful for our problem are

$$\boldsymbol{k} \cdot V_a = 0, \tag{16}$$

$$\Delta p_a = i \frac{\Delta \varrho_a}{\bar{\varrho}} \frac{d\bar{p}}{dr} \frac{k_r}{k^2}, \tag{17}$$

$$V_a = \frac{\Delta \varrho_a}{\bar{\varrho}^2} \frac{\tau}{\alpha} \left[\nabla p - (\boldsymbol{k} \cdot \nabla \bar{p}) \frac{\boldsymbol{k}}{k^2} \right] \tag{18}$$

$$\tau = \frac{l_r}{V_r}, \tag{19}$$

$$\boldsymbol{k} = \left(k_r, \pm \frac{k_r}{2}, \pm \frac{k_r}{2} \right). \tag{20}$$

l_r is the vertical mean free path and k_r the radial component of $\boldsymbol{k}$. The other components of $\boldsymbol{k}$ are such that convection be isentropic. Moreover, with the following choice of the constants

$$\alpha = \frac{2}{3} \qquad \mathscr{L} = \frac{l_r}{\sqrt{2\pi}} \tag{21}$$

the relations for static convections are then identical to those, given by Henyey *et al.* (1968), used to compute the stellar models.

IV. Linearized Equations for Time Dependent Convection

It is, in principle straightforward to perturb linearly Eqs. (6)–(14). However in practice a difficulty is immediately encountered since no rigorous solution is known for the unperturbed state where the Böhm-Vitense theory is used. Nevertheless, if we consider only perturbations whose characteristic lengths are large compared to those of convection, we may follow the same procedure as Unno (1967) (see Gabriel *et al.* 1974). With f having the same meaning as in Eq. (15) and δ being the symbol for lagrangian perturbations, we have:

$$\delta f = \delta f_a \exp[i \boldsymbol{k} \cdot \boldsymbol{r}] \tag{22}$$

with

$$\delta f_a \propto \exp[i \sigma t].$$

Within these limits the perturbations of Eqs. (6)–(8) write

$$\sum_i k_i \delta V_a^i = 0\,, \tag{23}$$

$$\begin{aligned}\frac{i\sigma\tau+\alpha}{\tau}\,\delta V_a^j &= \delta\left(\frac{\Delta\varrho_a}{\bar\varrho}\right)\frac{\nabla^j\bar p}{\bar\varrho} + \frac{\Delta\varrho_a}{\bar\varrho}\,\delta\left(\frac{\nabla^j\bar p}{\bar\varrho}\right)\\ &\quad - ik^j\left(\delta\,\frac{\Delta p_a}{\bar\varrho} - \frac{\delta\bar\varrho}{\bar\varrho}\,\frac{\Delta p_a}{\bar\varrho}\right)\\ &\quad + \frac{\alpha V^j}{\tau}\,\frac{\delta\tau}{\tau} - i\sigma(V\cdot\nabla)\,\delta r^j\,,\end{aligned} \tag{24}$$

$$\begin{aligned}&\frac{i\sigma\tau}{i\sigma\tau+\omega_R\tau+1}\,\frac{1+Q}{C_pQ}\,\delta\bar S + \frac{\delta\Delta S_a}{\Delta S_a}\\ &= \frac{1}{i\sigma\tau+\omega_R\tau+1}\left[\frac{\delta\tau}{\tau} - \omega_R\tau\,\frac{\delta\omega_R}{\omega_R}\right.\\ &\quad \left. + (\omega_R\tau+1)\left(\frac{\delta V_a^r}{V_a^r} + \sum_i \frac{V_a^i\,\delta V_i\bar S}{V_a^r\,\frac{d\bar S}{dr}}\right)\right].\end{aligned} \tag{25}$$

Now on we shall omit the subscript a since only the δf_a's will be considered.

We have also

$$\frac{\delta\Delta S}{\Delta S} = \frac{\delta(C_pQ)}{(C_pQ)} + \frac{\delta\left(\frac{\Delta\varrho}{\bar\varrho}\right)}{\frac{\Delta\varrho}{\bar\varrho}}\,, \tag{26}$$

$$\frac{\delta\tau}{\tau} = \frac{\delta l_r}{l_r} - \frac{\delta V_r}{V_r}\,, \tag{27}$$

$$\frac{\delta F_C^i}{F_C^i} = \left(\frac{\delta\bar\varrho}{\bar\varrho} + \frac{\delta\bar T}{\bar T}\right)\delta^{ir} + \frac{\delta V^i}{V^r} + \frac{\overline{V^i\delta\Delta S}}{\overline{\Delta S V^r}}\,, \tag{28}$$

where δ^{ij} is the Kronecker symbol.

Let us now multiply Eq. (24) by k_j and sum over j. Taking (23) into account $\delta\Delta p$ can be eliminated from (24). Then with the help of Eqs. (25) and (16) we eliminate $\delta\left(\frac{\Delta\varrho}{\bar\varrho}\right)$. This gives $\frac{\delta V^j}{V^r}$ in terms of perturbations of average quantities only.

Finally averaging over all possible vectors $\boldsymbol{k}$ for a given k_r we obtain, taking (18) into account:

$$\begin{aligned}&\left(\frac{i\sigma\tau+2\alpha}{\alpha} - \frac{\omega_R\tau}{i\sigma\tau+\omega_R\tau+1}\right)\overline{\frac{\delta V^r}{V^r}}\\ &= -\frac{\delta(C_pQ)}{(C_pQ)} + (i\sigma\tau+\omega_R\tau+1)^{-1}\\ &\quad\cdot\left[\frac{\delta l_r}{l_r} - i\sigma\tau\,\frac{1+Q}{C_pQ}\,\delta\bar S - \omega_R\tau\,\frac{\delta\omega_R}{\omega_R} + (\omega_R\tau+1)\frac{\delta\frac{d\bar S}{dr}}{\frac{d\bar S}{dr}}\right]\\ &\quad + \frac{\delta\left(\frac{1}{\varrho}\frac{dp}{dr}\right)}{\frac{1}{\varrho}\frac{dp}{dr}} + \frac{\delta l_r}{l_r} - \frac{i\sigma\tau}{3\alpha}\,\nabla\cdot\delta\boldsymbol{r}\end{aligned} \tag{29}$$

and for $j \neq r$

$$\begin{aligned}&\frac{i\sigma\tau+\alpha}{\alpha}\,\overline{\frac{\delta V^j}{V^r}}\\ &= \frac{\omega_R\tau+1}{i\sigma\tau+\omega_R\tau+1}\,\frac{\delta(\nabla^j S)}{\frac{dS}{dr}} + \frac{5}{2}\,\frac{\delta\left(\frac{\nabla^j p}{\varrho}\right)}{\frac{1}{\varrho}\frac{dp}{dr}}\\ &\quad - \frac{i\sigma\tau}{\alpha}\left[\frac{5}{6}\,\nabla_r\delta r^j + \frac{1}{3}\,\nabla^j\delta r_r\right]\\ &\quad + \left(\frac{\omega_R\tau}{i\sigma\tau+\omega_R\tau+1} - 1\right)\frac{\overline{V^j\delta V^r}}{\overline{V^r V^r}}\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\overline{\delta\Delta S}}{\overline{\Delta S}} &= \frac{1}{i\sigma\tau+\omega_R\tau+1}\left[\frac{\delta l_r}{l_r} - \omega_R\tau\left(\frac{\delta\omega_R}{\omega_R} - \overline{\frac{\delta V^r}{V^r}}\right)\right.\\ &\quad\left. + (\omega_R\tau+1)\,\frac{\delta\frac{d\bar S}{dr}}{\frac{d\bar S}{dr}} - i\sigma\tau\,\frac{1+Q}{C_pQ}\,\delta\bar S\right],\end{aligned} \tag{30}$$

$$\begin{aligned}&\left(\frac{i\sigma\tau+2\alpha}{\alpha} - \frac{\omega_R\tau}{i\sigma\tau+\omega_R\tau+1}\right)\frac{\overline{V^j}}{V^r}\,\frac{\overline{\delta V^r}}{V^r} = \frac{\delta\left(\frac{\nabla^j p}{\varrho}\right)}{\frac{1}{\varrho}\frac{dp}{dr}}\\ &\quad + \frac{\omega_R\tau+1}{i\sigma\tau+\omega_R\tau+1}\,\frac{\delta(\nabla^j\bar S)}{\frac{d\bar S}{dr}} - \frac{i\sigma\tau}{3\alpha}(\nabla^j\delta r_r + \nabla_r\delta r^j).\end{aligned} \tag{31}$$

It can easily be verified that

$$\frac{\overline{V^r\delta(\Delta S)}}{\overline{V^r\Delta S}} = \frac{\overline{\delta\Delta S}}{\overline{\Delta S}} \tag{32}$$

and for $j \neq r$

$$\frac{\overline{V^j\delta(\Delta S)}}{\overline{V^r\Delta S}} = \frac{\omega_R\tau+1}{i\sigma\tau+\omega_R\tau+1}\left(\frac{\delta\nabla^j\bar S}{\frac{d\bar S}{dr}} + \frac{\overline{V^j}}{V^r}\,\frac{\overline{\delta V^r}}{V^r}\right) \tag{33}$$

$$\sum_j \frac{\overline{V_j\delta V^j}}{V^2} = \frac{\overline{\delta V^r}}{V^r}\,. \tag{34}$$

The perturbation of the mean free path l_r cannot of course be given by the theory. In order to obtain the same expressions for radial perturbations when $\sigma\tau \ll 1$ as those deduced from the equations for stationary convection, we adopt

$$\frac{\delta l_r}{l_r} = \frac{\delta H_p}{H_p}\,. \tag{35}$$

It should be pointed out that when the equations for stationary convection are perturbed special care must be taken in order to write them in the proper form. For instance consider Eq. (29) for $\sigma\tau = 0$ and $\omega_R\tau = 0$

(adiabatic motion of turbulent eddies) in the case of radial oscillations. It becomes

$$2\frac{\delta V^r}{V^r}=2\frac{\delta l_r}{l_r}+\frac{\delta\frac{d\bar{S}}{dr}}{\frac{d\bar{S}}{dr}}-\frac{\delta(C_pQ)}{C_pQ}+\frac{\delta\left(\frac{1}{\varrho}\frac{dp}{dr}\right)}{\frac{1}{\varrho}\frac{dp}{dr}}$$

or

$$2\frac{\delta V^r}{V^r}=\frac{\delta l_r}{l_r}+\frac{\delta\left(\frac{\Delta\varrho}{\bar{\varrho}}\right)}{\frac{\Delta\varrho}{\bar{\varrho}}}+\frac{\delta\left(\frac{1}{\varrho}\frac{dp}{dr}\right)}{\frac{1}{\varrho}\frac{dp}{dr}}$$

since when $\omega_R\tau=0$ $\Delta S=C_pQ\frac{\Delta\varrho}{\varrho}=\frac{d\bar{S}}{dr}l_r$.

This is also the perturbation of the radial component of (18) which writes

$$V_r^2=\frac{1}{3\alpha}\frac{\Delta\varrho}{\bar{\varrho}}\frac{1}{\varrho}\frac{dp}{dr}l_r.$$

But the term $\frac{1}{\varrho}\frac{dp}{dr}$ may not be replaced by $-\frac{Gm}{r^2}$.

Equations (29)–(34) expressed in spherical coordinates allow the computation of δF_C^i.

V. Coefficient of Vibrational Stability

If the damping is such that

$$\delta r\propto\exp[-\sigma' t]. \tag{36}$$

σ' is given by (Ledoux and Walraven, 1958)

$$\sigma'=-\frac{1}{2\sigma^2}\frac{\int_0^{M_a}\frac{\delta T}{T}\delta\left[\varepsilon-\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right]dm}{\int_0^M(\delta r)^2\,dm} \tag{37}$$

with

$$\int_0^M(\delta r)^2\,dm=\int_0^M\left[\delta r^2+\frac{l(l+1)}{\sigma^4}\frac{\chi^2}{r^2}\right]dm. \tag{38}$$

M_a indicates the value of m where the adiabatic approximation breaks down.

It should be remembered that in Eqs. (37) and (38) only the spatial dependence of the perturbations must be taken into account.

Ledoux (1974) has given an expression for $\delta\left(\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right)$ valid in a radiative zone. Here we generalize his formula to include the contribution of the convective flux.

Keeping in mind that

$$\delta F^r=\delta F^r(r)\,Y_l^m$$

$$\delta F^\theta=\frac{\delta F_\theta(r)}{r^2}\frac{\partial Y_l^m}{\partial\theta}\qquad \delta F^\varphi=\frac{\delta F_\theta(r)}{r^2\sin^2\theta}\frac{\partial Y_l^m}{\partial\varphi}$$

we have for the perturbed radiative flux F_R

$$\frac{\delta F_R^r(r)}{F_R^r}=\left(4-\frac{\partial\ln\varkappa}{\partial\ln T}\right)\frac{\delta T}{T}-\left(1+\frac{\partial\ln\varkappa}{\partial\ln\varrho}\right)\frac{\delta\varrho}{\varrho}+\frac{d\frac{\delta T}{T}}{d\ln T}-\frac{d\delta r}{dr}, \tag{39}$$

$$\frac{\delta F_\theta(r)}{F_R^r}=\frac{\delta T}{T}\left(\frac{d\ln T}{dr}\right)^{-1}-\left(\delta r-\frac{\chi}{\sigma^2 r}\right). \tag{40}$$

Taking into account the differential equation for spherical surface harmonics we obtain

$$\begin{aligned}\varrho\delta\left(\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right)=&-\frac{1}{r^2}\frac{d}{dr}(r^2F^r)\frac{\delta\varrho}{\varrho}\\&+\frac{1}{r^2}\frac{d}{dr}(r^2\delta F^r)-\frac{dF}{dr}\frac{d\delta r}{dr}\\&-\frac{F^r}{r}\left[2\frac{\delta r}{r}-l(l+1)\frac{\chi}{\sigma^2r^2}\right]\\&-l(l+1)\frac{\delta F_\theta}{r^2}\end{aligned} \tag{41}$$

with

$$\begin{aligned}\delta F^r&=\delta F_R^r(r)+\delta F_C^r(r)\\ \delta F_\theta&=\delta F_{R,\theta}(r)+\delta F_{C,\theta}(r).\end{aligned} \tag{42}$$

The equation of continuity and the radial component of the equation of motion may be used to write Eq. (41) in the following way:

$$\delta\left(\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right)=\frac{d\delta L^r}{dm}-\frac{l(l+1)}{r^2}\left[\frac{\delta F_\theta}{\varrho}+\frac{\chi}{\sigma^2}\left(\frac{dL}{dm}-\frac{F}{\varrho r}\right)\right] \tag{43}$$

where

$$\frac{\delta L^r}{L^r}=2\frac{\delta r}{r}+\frac{\delta F^r}{F^r}. \tag{44}$$

If we introduce

$$\frac{\delta L_1^r}{L^r}=\frac{\delta L^r}{L^r}+l(l+1)\frac{\chi}{\sigma^2r^2}.$$

Equation (43) becomes

$$\delta\left(\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right)=\frac{d\delta L_1^r}{dm}-\frac{l(l+1)}{r^2}\left[\frac{\delta F_\theta}{\varrho}+\frac{2\chi}{\sigma^2}\frac{dL}{dm}+\frac{F^r}{\sigma^2\varrho}\left(\sigma^2\delta r-A\frac{\delta p}{\varrho}-3\frac{\chi}{r}\right)\right] \tag{45}$$

with

$$A=\frac{d\ln\varrho}{dr}-\frac{1}{\Gamma_1}\frac{d\ln p}{dr}.$$

This last expression shows up the destabilizing effect discovered by Souffrin and Spiegel (1967).

It corresponds to the term

$$\frac{l(l+1)}{r^2}\frac{F^2 A}{\sigma^2 \varrho}\frac{\delta p}{\varrho}$$

which is stabilizing in a radiative zone ($A<0$) and destabilizing in a convective zone with a superadiabatic gradient ($A>0$).

VI. Comparison with Previous Results

Comparison will be limited to the case of radial oscillations since we do not know of any published work on the non radial case. We will moreover assume that when $\sigma\tau \gg 1$, $\omega_R \tau \gg 1$ and $\delta \bar{S}=0$.

Cox's theory as well as ours impose that for $\sigma\tau \ll 1$ the results must be identical to those deduced from the equations for static convection; they are therefore in good agreement in that limiting case. On the other hand, when $\sigma\tau \gg 1$ Cox imposes $\lim_{\sigma\tau\to\infty} \delta F_C^r = 0$. A much better agreement between the theories would be achieved if such a condition were applied to $\delta(F_C^r/\varrho T) = \delta(\overline{\Delta S V})$ instead of δF_C^r.

If it is obvious that ΔS and V may undergo phase shifts with respect to the pulsation, we do not see how this could be justified for ϱ and T.

With these latter hypotheses, Cox's theory would give essentially the same results as these of Boury *et al.* when $\sigma\tau \gg 1$. They would still differ from ours, in two respects.

First we obtain $\delta\Delta S=0$ while they have $\delta\left(\frac{\Delta\varrho}{\varrho}\right)\Big/\left(\frac{\Delta\varrho}{\varrho}\right) = \frac{\delta\Delta S}{\Delta S} - \frac{\delta C_p Q}{C_p Q} = 0$. However in most cases $\delta(C_p Q)$ is small. On the other hand we obtain $\frac{\delta V^r}{V^r} = \frac{1}{3}\frac{\delta\varrho}{\varrho}$ instead of $\delta V^r/V^r = 0$. This is because we have kept the term $(\Sigma\, \varrho V^i V_i u^j)$ in Eq. (7). It corresponds to the term $\frac{1}{3}\frac{V^2}{\varrho}\frac{d\varrho}{dt}$ in the equation of conservation of kinetic energy of turbulence

$$\frac{1}{2}\frac{d\overline{V^2}}{dt} - \frac{1}{3}\frac{\overline{V^2}}{\bar{\varrho}}\frac{d\bar{\varrho}}{dt} = -\frac{\alpha\overline{V^2}}{\tau} - \frac{1}{\varrho}\overline{V\cdot \nabla p} \tag{46}$$

whose perturbation writes (taking (34) into account)

$$\frac{i\sigma\tau}{\alpha}\left(\frac{\delta V^r}{V^2} - \frac{1}{3}\frac{\delta\varrho}{\varrho}\right) = \frac{\delta(\overline{V^2}/\tau)}{\overline{V^2}/\tau} - \frac{\delta\left(\frac{1}{\varrho}\overline{V\cdot\nabla p}\right)}{\frac{1}{\varrho}\overline{V\cdot\nabla p}}. \tag{47}$$

This shows that when $\sigma\tau \gg 1$ the perturbation of the kinetic energy of turbulence results only from the dilatation work done to overcome the turbulent pressure $p_t = \frac{1}{3}\varrho\overline{V^2}$. When $\sigma\tau \ll 1$ Eq. (47) gives the same results as when Eq. (22) is perturbed. Keeping $\Sigma\, \varrho V^i V_i u^j$ in the equation of motion (7) leads to results which are coherent with these deduced from Eqs. (46) and (47).

When $\sigma\tau \ll 1$ Boury *et al.* have results very different from all others. This is because they have supposed that the perturbations of V^r may always be computed using the adiabatic approximation (i.e. equating the left side member to zero) of Eq. (46). Our results and those of Unno show that this hypothesis is not allowed.

Our results are very close to Unno's. They differ on 2 points. First Unno neglects the term $\Sigma\, V^i V_i u^j$ in the equation of motion. Secondly we feel that he eliminates too quickly the enthalpy from the equation of energy conservation. This, along with the approximations mentioned above leads, when $\sigma\tau \gg 1$, to $\delta\Delta T=0$ while we have $\delta\Delta S = \delta\left(C_p\frac{\Delta T}{T}\right)=0$. Our result is supported by an intuitive picture: Suppose $\sigma\tau \gg 1$, then the displacement of a bubble during a period is very small, we may even consider that it does not move if $\sigma\tau=\infty$.

For an adiabatic perturbation the entropies of the bubble and that of the surrounding material remain unchanged, leading to $\delta\Delta S=0$.

VII. General Discussion of the Influence of Convection on Vibrational Stability

This theory will be applied in studies of vibrational stability towards non radial oscillations [for the first results see Gabriel *et al.* (1974) and Noels *et al.* (1974)]. We shall give here only a general discussion in the case of $30 M_\odot$ models as an example of stars with a convective core and in the case of solar models as an example of stars with a convective envelope.

In a convective core $\sigma\tau \gg 1$ and $\omega_R\tau \ll 1$. With a good accuracy we have then

$$\frac{\delta V^r}{V^r} = \frac{1}{3}\frac{\delta\varrho}{\varrho}$$

$$\frac{\delta V_\theta(r)}{V^r} = -\frac{7}{6}\left(\delta r - \frac{\chi}{\sigma^2 r}\right)$$

$$\frac{\delta\Delta S}{\Delta S} = 0$$

$$\frac{\delta F_C^r}{F_C^r} = \frac{4}{3}\frac{\delta\varrho}{\varrho} + \frac{\delta T}{T} \qquad \frac{\delta F_{C,\theta}}{F_C^r} = \frac{\delta V_\theta(r)}{V^r}.$$

For the fundamental and the first p and g^+ modes of $l=1$ and 2, the radial component of the perturbed convective luminosity has a destabilizing influence near the center. This is because, at $r=0$, $\frac{\delta T}{T}=0$ but $\frac{\delta r}{r} \neq 0$ ($l=1$ or 2). Further out however it gets a

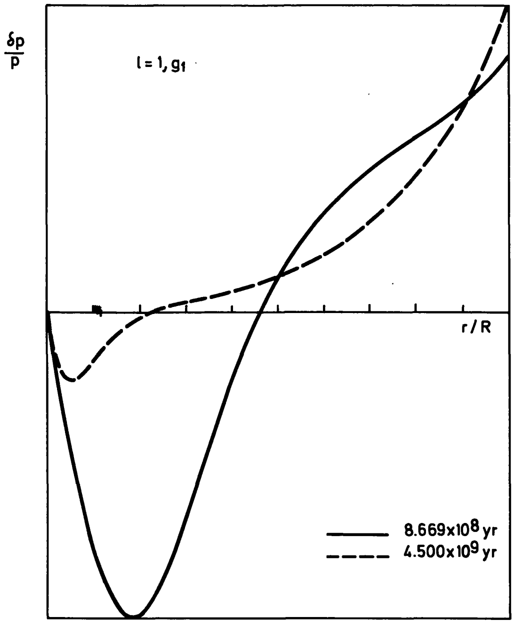

Fig. 1. $\delta p/p$ as function of r/R for the g_1^+ mode ($l=1$) of a solar model 8.67 10^8 years old (full line) and of the present sun (dotted line)

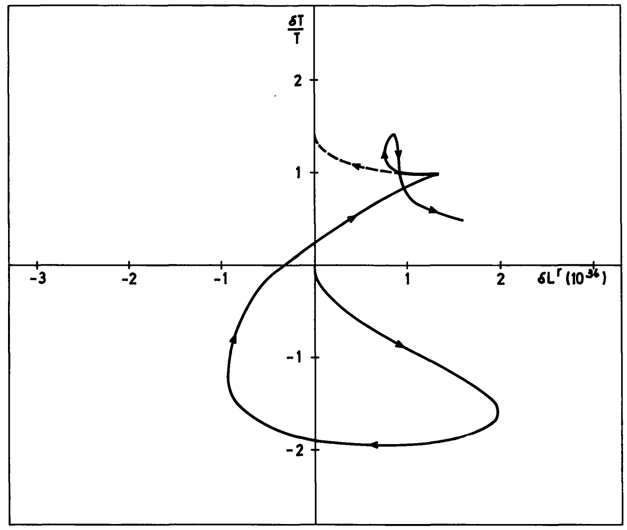

Fig. 2. $\delta T/T$ as function of δL^r for the g_1^+ model ($l=1$) for a solar model 8.67 10^8 years old

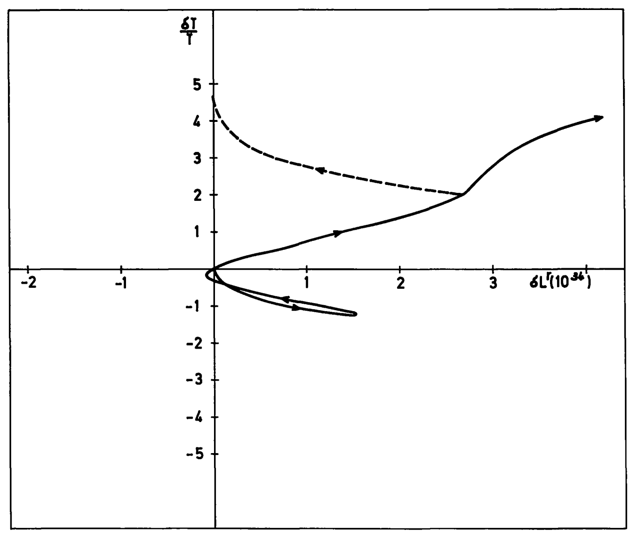

Fig. 3. $\delta T/T$ as function of δL^r for the g_1^+ mode ($l=1$) of the present sun

stabilizing influence until L_C decreases near the surface of the convective core. In these layers it has a destabilizing influence. As a whole the contribution of δL^r_C in the convective core is stabilizing.

For the modes we considered, the non radial component of the convective flux has a destabilizing influence everywhere in the core.

If we split δL^r and δF_θ in their radiative and convective parts, in Eq. (44), $\delta\left(\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right)$ contributes to the numerator of σ' by 6 terms which are all of the same order of magnitude. This means that the influence of convection may be neglected only if the contribution of the convective core (of mass M_C) to the numerator of σ' is small compared to that of the whole star. For the $30\,M_\odot$ models the ratio

$$R=\frac{\int_0^{M_C}\frac{\delta T}{T}\,\delta\left(\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right)dm}{\int_0^{M_a}\frac{\delta T}{T}\,\delta\left(\frac{1}{\varrho}\nabla\cdot\boldsymbol{F}\right)dm}$$

is always smaller than 10^{-2} and convection could be neglected. This result depends essentially upon the behaviour of the perturbations thoughout the star. If they are much smaller in the convective core than outside, R will be small and so will be the role played by the convective core on the stability. The more the amplitudes grow in the core compared the outer layers, the more R increases; the role of convection becomes more and more important.

In the convective envelope of $1\,M_\odot$ models, we still have $\sigma\tau \gg 1$ and $\omega_R\tau < 1$ except in the upper layers of the envelope (where the adiabatic approximation is no longer valid). Therefore the non radial component of the perturbation of the convective flux has a destabilizing influence in the major part of the envelope.

The change of sign of $\delta F_{\theta,C}$ occurs when $\omega_R\tau \simeq 0.5$.

The ratio of the contributions of the non radial component of the convective flux to the radial one is smaller than 5 10^{-2} for the first p and g^+ modes. This is because the horizontal characteristic length of the perturbations $[l(l+1)/r^2]^{\frac{1}{2}}$ is much larger than the vertical scale height $\left(\frac{d\ln\delta T/T}{dr}\right)^{-1}$. The influence of the convective flux on the term $\int\frac{\delta T}{T}\,d(\delta L^r_R+\delta L^r_C)$

Table 1

	$\int \frac{\delta T}{T}\, d\,\delta L^r$		$\int \frac{\delta T}{T}\, \delta\left(\varepsilon - \frac{1}{\varrho}\nabla \cdot \boldsymbol{F}\right) dm$		$(\sigma')^{-1}$ years	
	$\delta L^r_C = 0$	$\delta L^r_C \neq 0$	$\delta F^i_C = 0$	$\delta F^i_C \neq 0$	$\delta F^i_C = 0$	$\delta F^i_C \neq 0$
1	2.781 (34)	4.244 (34)	2.964 (34)	1.523 (34)	– 1.300 (7)	– 2.494 (7)
SUN	– 5.412 (34)	2.467 (35)	6.741 (34)	– 2.297 (35)	– 6.232 (6)	4.612 (5)

is shown in Figs. 2 and 3 which give δT in term of $(\delta L^r_R + \delta L^r_C)$. The dotted curves are obtained with the assumption that $\delta L^r_C = 0$. We see that this hypothesis has a destabilizing influence, all the larger as $\delta T/T$ is large in the outer layers compared to its inner values. Figure 2 refers to the g_1^+ mode $(l=1)$ of a sun model which is 8.6710^8 years old (hereafter called model 1) and whose $\frac{\delta p}{p}$ is given by the full curve in Fig. 1.

The equivalent curves in the case of the present sun are given in Fig. 3. The corresponding eigenfunction $\delta p/p$ is represented by the dotted line in Fig. 1. The values of $\int \frac{\delta T}{T}\, d\,\delta L$, of the numerator of σ' and of σ' are given in Tables 1 in 2 cases: first taking the perturbation of the convective flux into account (as proposed here) second neglecting it.

We see that for model 1, $\frac{\delta p}{p}$ is larger at $x \simeq 0.25$ than in the outer convective envelope $(x > 0.948)$. This region will have the largest weight in the integral giving σ'. The outer layers have nevertheless a non negligible influence, it is shown by the factor of 2 in σ' between the 2 cases. The model is anyway found unstable.

For the present sun, $\frac{\delta p}{p}$ is much larger in the convective envelope $(x > 0.941)$ than in the interior. Consequently, σ' is much more sensitive to the perturbations of the convective flux. Figure 3 shows that if $\delta L^r_C = 0$, $\int_0^m \frac{\delta T}{T}\, d\,\delta L$ changes its sign as soon as the upper limit of integration m slightly exceeds the mass at the bottom of the convective envelope. With $\delta L^r_C \neq 0$ it grows up to the point where the adiabatic approximation breaks down.

The model is found unstable with $\delta L^r_C = 0$ and stable with $\delta L^r_C \neq 0$. Such a situation will be encountered every time that the amplitude in the convective envelope is larger than in the radiative core. Neglecting convection in such cases would lead to meaningless results.

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