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Vibrational Instability of a 1 $M_\odot$ Star towards Non-radial Oscillations

A. Boury, M. Gabriel, A. Noels, R. Scuflaire and P. Ledoux
Institut d'Astrophysique de l'Université de Liège

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Summary. The vibrational stability for non-radial oscillations of a 1 $M_\odot$ star during the main sequence phase, is studied. The star becomes unstable at an age of 2.4×10^8 years. Stability is restored after three billion years.

Key words: non-radial oscillations — vibrational stability — solar neutrinos

I. Introduction

In an attempt to explain the apparent absence of solar neutrinos, Fowler (1972) proposed that the sun could have undergone, 10^7 years ago, a change in structure by sudden mixing. The arrival of fresh hydrogen and helium-3 at the solar centre causes a decrease in temperature, slowing down nuclear reactions and sharply reducing the neutrino emission. Ezer and Cameron (1972) and Rood (1972) confirmed this by making detailed calculations of the effect of a sudden mixing on the subsequent evolution of the Sun. It remains then to find a mechanism capable of inducing the transient mixing desired. Dilke and Gough (1972) suggested that the Sun is unstable towards the low order gravity modes of non-radial pulsation, because of the presence of a gradient in hydrogen and helium-3 abundances. Dziembowski and Sienkewicz (1973) calculated the stability of a solar model; they found it stable but their result was open to question since they had used the static value of the temperature sensitivity of the nuclear energy generation rate, ε, whereas it is the high value of the effective response of ε to temperature fluctuations which favours instability. With this taken properly into account, Christensen-Dalsgaard and Gough (1974), Christensen-Dalsgaard *et al.* (1974), and Noels *et al.* (1974) found that the present Sun is indeed stable but that instability occurs in earlier phases of its evolution. We give here detailed results of our analysis which we carried out using the lagrangian formalism. In Section II, the principal properties of the models tested for stability are described. A brief exposé of the method of calculating adiabatic non-radial oscillations is given in Section III. The stability analysis is made in Section IV.

II. Models

The evolution of a 1 $M_\odot$ star of initial chemical composition $X=0.7417$, $Y=0.2383$ from the gravitational contraction to the main sequence phase was computed by the Henyey method. The ratio of the mixing length to the pressure scale height was chosen to be 1.55, in order to yield the present value of the solar radius. The age of the Sun was taken as 4.5×10^9 years. The opacities were obtained by interpolation in Cox and Stewart's tables (1970). The rates of the nuclear reactions were those given by Bahcall and May (1969) for the proton-proton reaction, Dwarakanath and Winkler (1971) for the ^{3}He-^{3}He reaction, Fowler *et al.* (1974) for the ^{16}O-^{1}H reaction and Fowler *et al.* (1967) for all others in the proton-proton chain and the C–N–O cycle. We give, in Table 1, the properties of the models tested for vibrational stability. Model 1 nearly corresponds to the zero-age main sequence while model 4 is closest to the present Sun.

III. Non-radial Adiabatic Oscillations

The theory of non-radial oscillations can be found in Ledoux and Walraven (1958). We neglected the non-adiabatic terms and integrated numerically the fourth-order differential system, taking into account the perturbation of the gravitational potential. All amplitudes are written as the product of a function

Table 1. Properties of the models

Model Number	Age (years)	X_c	T_c	ϱ_c	$\varrho_c/\bar{\varrho}$	Log T_e	L
1	1.04(8)	0.7363	1.329(7)	85.62	43.68	3.7492	2.746(33)
2	8.67(8)	0.6806	1.348(7)	93.45	50.91	3.7508	2.914(33)
3	2.40(9)	0.5651	1.406(7)	111.9	68.38	3.7543	3.238(33)
4	4.50(9)	0.3930	1.513(7)	153.2	110.4	3.7595	3.822(33)
5	5.33(9)	0.3181	1.569(7)	177.2	141.0	3.7613	4.115(33)

Numbers in parentheses indicate the power of 10 which multiplies the preceding numbers.

of the radial distance, r, times $P_l^m(\cos\theta)\exp(im\varphi)$ where P_l^m is the associated Legendre polynomial.
Let

$$x=r/R\,, \tag{1}$$

$$y=x^{1-l}\delta r/R\,, \tag{2}$$

$$z=x^{-l}\delta p/p\,, \tag{3}$$

$$u=x^{-l}R\Phi'/GM\,, \tag{4}$$

$$v=x^{1-l}\left(\frac{R^2}{GM}\frac{d\Phi'}{dr}+\frac{4\pi\varrho R^3}{M}\frac{\delta r}{R}\right), \tag{5}$$

where l, δr, δp and Φ' are the degree of the spherical surface harmonic, the radial displacement, the lagrangian perturbation of the pressure, and the eulerian perturbation of the gravitational potential, respectively. Other symbols are conventional.
The differential system (Section 75 in Ledoux and Walraven, 1958) takes the form

$$\frac{dy}{dx}=\frac{l+1}{x}\left[-y+\frac{l}{\omega^2}\left(\frac{qy}{x^3}+u+\frac{Rp}{GM\varrho}z\right)\right]-\frac{xz}{\Gamma_1}, \tag{6}$$

$$\frac{dz}{dx}=\frac{GM\varrho}{Rp}\left[\left(\omega^2+4\frac{q}{x^3}\right)\frac{y}{x}+\frac{q}{x^2}z-\frac{v}{x}-\frac{l(l+1)}{\omega^2}\frac{q}{x^4}\cdot\left(\frac{q}{x^3}y+\frac{Rp}{GM\varrho}z+u\right)\right]-\frac{lz}{x}, \tag{7}$$

$$\frac{du}{dx}=\frac{1}{x}\left(v-\frac{4\pi R^3\varrho}{M}y-lu\right), \tag{8}$$

$$\frac{dv}{dx}=\frac{l(l+1)}{x}(lu-v)+\frac{l(l+1)}{x\omega^2}\frac{4\pi R^3\varrho}{M}\left(\frac{qy}{x^3}+\frac{Rpz}{GM\varrho}+u\right), \tag{9}$$

where $q=m(r)/M$ and $\omega^2=R^3\sigma^2/GM$, σ being the angular frequency. At the centre, the regularity of the solution implies

$$\omega^2 y=l\left(\frac{qy}{x^3}+\frac{Rpz}{GM\varrho}+u\right), \tag{10}$$

$$v=\frac{4\pi R^3\varrho y}{M}+lu\,. \tag{11}$$

The continuity of the gravitational potential and of its first derivative through the surface of the star is expressed by

$$v+(l+1)u=0 \tag{12}$$

$$\sigma'_{k,l}=-\frac{1}{2\sigma^2_{k,l}}\,\frac{\int_0^{M_a}\left(\frac{\delta T}{T}\right)_{k,l}\delta\left(\varepsilon-\frac{1}{\varrho}\nabla\cdot F\right)_{k,l}dm+\int_0^{M_a}\left(\Gamma_3-\frac{5}{3}\right)\left(\frac{\delta\varrho}{\varrho}\right)_{k,l}\delta\left(\varepsilon_2+\frac{1}{\varrho}\bar{\nabla}\cdot\nabla p\right)_{k,l}dm}{\int_0^M|\delta r|^2_{k,l}dm}\,. \tag{14}$$

δp must vanish at the surface. This condition may be written, from Eq. (7)

$$z+\left[4+\omega^2-\frac{l(l+1)}{\omega^2}\right]y-\frac{l(l+1)u}{\omega^2}-v=0\,. \tag{13}$$

Table 2. Periods of adiabatic oscillation and vibrational stability results (see text)

	Model Number	P(s)	E_N	E_F	σ'^{-1} (years)[a]
$l=1\,g_1$	1	6.373(3)	1.261(35)	1.351(35)	9.153(7)
	2	5.983(3)	9.674(34)	7.555(34)	−1.851(7)
	3	5.096(3)	5.093(34)	3.357(34)	−7.209(6)
	4	3.845(3)	3.503(34)	1.404(35)	1.006(6)
	5	3.701(3)	2.303(34)	1.534(37)	4.182(4)
$l=1\,g_2$	1	9.679(3)	1.830(35)	2.059(35)	2.420(7)
	2	8.250(3)	5.602(34)	4.858(34)	−2.115(7)
	3	6.761(3)	1.718(34)	1.505(34)	−2.289(7)
	4	5.238(8)	1.254(34)	1.354(34)	3.858(7)
	5	4.669(3)	1.369(34)	1.922(34)	7.433(6)
$l=1\,g_3$	1	1.183(4)	9.987(34)	1.294(35)	7.292(6)
	2	1.047(4)	2.049(34)	2.118(34)	5.734(7)
	3	8.530(3)	6.379(33)	7.833(33)	1.258(7)
	4	6.556(3)	4.589(33)	6.739(33)	7.351(6)
	5	5.872(3)	5.073(33)	8.084(33)	5.701(6)
$l=2\,g_1$	1	4.370(3)	5.122(33)	2.556(34)	3.100(6)
	3	3.890(3)	4.729(32)	2.538(33)	8.535(5)
	4	3.367(3)	4.451(31)	1.976(33)	1.315(5)
$l=2\,g_2$	1	5.821(3)	3.452(33)	7.376(33)	4.552(6)
	3	4.694(3)	1.073(32)	1.891(32)	3.975(6)
	4	3.845(3)	1.918(31)	3.160(32)	2.942(5)
$l=2\,g_3$	3	5.605(3)	3.225(31)	5.195(31)	4.984(6)
	4	4.425(3)	7.888(30)	4.479(31)	9.117(5)

[a]) A negative sign means instability.

One then obtains by iteration the eigenvalue ω^2 for which a linear combination of two independent solutions, satisfying conditions (10) and (11) also fulfils conditions (12) and (13).
The periods, P, of the modes g_1, g_2 and g_3, $l=1,2$ are listed in Table 2. To illustrate the behaviour of the corresponding eigenfunctions, we have plotted in Fig. 1 and 2, for the modes g_1, g_2, $l=1$ the distribution of $\delta p/p$ in all five models.

IV. Vibrational Stability

With a time dependence of the perturbations of the form $\exp(i\sigma_{k,l}t-\sigma'_{k,l}t)$, the damping coefficient $\sigma'_{k,l}$ relative to the k mode associated with the lth harmonic writes (Section 65 and 81 in Ledoux and Walraven, 1958)

The non-adiabatic terms are, as usual, written in terms of the adiabatic solution. The integrals in the numerator are carried up to the value M_a of the mass where the adiabatic approximation breaks down, that is, where the non-adiabatic correction to $\delta T/T$ becomes of the order of the adiabatic perturbation (Ledoux, 1965).

The amplitude of the perturbation of the nuclear energy generation, $\delta\varepsilon$, is expressed as (Ledoux, 1965)

$$\delta\varepsilon=\sum_{i,j}\varepsilon_{i,j}\left(\mu_{i,j}\frac{\delta\varrho}{\varrho}+\nu_{i,j}\frac{\delta T}{T}+\frac{\delta X_i}{X_i}+\frac{\delta X_j}{X_j}\right) \tag{15}$$

where we have dropped the subscripts k, l for brevity. The sum is carried over all reactions (i, j) taking place. X_i and X_j represent the relative mass abundances of the reagents in reaction (i, j), and

$$\mu_{i,j}=\left(\frac{\partial\log\varepsilon_{i,j}}{\partial\log\varrho}\right)_{X_i,X_j,T}, \quad \nu_{i,j}=\left(\frac{\partial\log\varepsilon_{i,j}}{\partial\log T}\right)_{X_i,X_j,\varrho}. \tag{16}$$

The $(\delta X_i/X_i)$'s are calculated by solving the kinetic equations pertinent to the proton-proton chain and to the carbon cycle (Schatzman, 1951). The latter, however, accounts for only five per cent of the energy generated at the centre of our models.

In the proton-proton chain, all intermediary isotopes have achieved their equilibrium abundances: in model 1, ^{3}He has reached equilibrium in all layers where ε is larger than one tenth its central value.

One then finds:

$$\begin{aligned}
&\frac{\delta X_{^2\mathrm{H}}}{X_{^2\mathrm{H}}}=\frac{(K_{^2\mathrm{H},^1\mathrm{H}}\varrho X_{^1\mathrm{H}})^2}{(K_{^2\mathrm{H},^1\mathrm{H}}\varrho X_{^1\mathrm{H}})^2+\sigma^2}(\nu_{^1\mathrm{H},^1\mathrm{H}}-\nu_{^2\mathrm{H},^1\mathrm{H}})\frac{\delta T}{T}\\
&\frac{\delta X_{^3\mathrm{He}}}{X_{^3\mathrm{He}}}\simeq 0 \quad \frac{\delta X_{^7\mathrm{Be}}}{X_{^7\mathrm{Be}}}\simeq 0\\
&\frac{\delta X_{^7\mathrm{Li}}}{X_{^7\mathrm{Li}}}=\frac{(K_{^7\mathrm{Li},^1\mathrm{H}}\varrho X_{^1\mathrm{H}})^2}{(K_{^7\mathrm{Li},^1\mathrm{H}}\varrho X_{^1\mathrm{H}})^2+\sigma^2}\\
&\qquad\cdot\left[\left(-\tfrac{1}{2}-\nu_{^7\mathrm{Li},^1\mathrm{H}}\right)\frac{\delta T}{T}-(\mu_{^7\mathrm{Li},^1\mathrm{H}}-1)\frac{\delta\varrho}{\varrho}\right]\\
&\frac{\delta X_{^8\mathrm{B}}}{X_{^8\mathrm{B}}}=\frac{\delta X_{^8\mathrm{Be}}}{X_{^8\mathrm{Be}}}=\left(\mu_{^7\mathrm{Be},^1\mathrm{H}}\frac{\delta\varrho}{\varrho}+\nu_{^7\mathrm{Be},^1\mathrm{H}}\frac{\delta T}{T}\right)
\end{aligned} \tag{17}$$

where $(K_{^2\mathrm{H},^1\mathrm{H}}\varrho X_{^1\mathrm{H}})^{-1}$ and $(K_{^7\mathrm{Li},^1\mathrm{H}}\varrho X_{^1\mathrm{H}})^{-1}$ are the mean lifetimes of ^{2}H and ^{7}Li.

In the carbon cycle, the δX_i relative to the long-lived isotopes are practically zero. Those of the radioactive isotopes whose mean lives are much shorter than the period of oscillations, are given by

$$\frac{\delta X_i}{X_i}=\left(\mu_{i-1}\frac{\delta\varrho}{\varrho}+\nu_{i-1}\frac{\delta T}{T}\right) \quad (i={}^{13}\mathrm{N},{}^{15}\mathrm{O}) \tag{18}$$

where the subscript $i-1$ refers to the preceding reaction in the cycle. Equation (15) may be written

$$\delta\varepsilon\equiv\varepsilon\left(\mu_e\frac{\delta\varrho}{\varrho}+\nu_e\frac{\delta T}{T}\right) \tag{19}$$

defining the effective sensitivity of the energy generation to density and temperature oscillations. μ_e is not very different from 1 but ν_e is much larger than the static value ν used in the expression $\varepsilon\propto\varrho^\mu T^\nu$.

In our models, ν is close to $\nu_{^1\mathrm{H},^1\mathrm{H}}$ as a consequence of the strong dominance of the proton-proton chain in the energy production.

Let us, for the purpose of discussion, consider the proton-proton chain alone. If one assumes for a moment that the chain always terminates through the ^{3}He-^{3}He reaction, then with

$$\frac{\delta X_{^2\mathrm{H}}}{X_{^2\mathrm{H}}}\simeq(\nu_{^1\mathrm{H},^1\mathrm{H}}-\nu_{^2\mathrm{H},^1\mathrm{H}})\frac{\delta T}{T} \tag{20}$$

one finds that

$$\nu_e\simeq(13.36\nu_{^1\mathrm{H},^1\mathrm{H}}+12.85\nu_{^3\mathrm{He},^3\mathrm{He}})/26.21 \tag{21}$$

where the numerical factors are the yields, in MeV, of the corresponding reactions.

At the centre of model 4, corresponding to the present Sun.

$$\nu_{^1\mathrm{H},^1\mathrm{H}}\simeq 4, \quad \nu_{^3\mathrm{He},^3\mathrm{He},}\simeq 16, \quad \nu_e\simeq 10.$$

If, on the contrary, the chain terminates through the ^{3}He-^{4}He reaction, then, as $\delta X_{^7\mathrm{Li}}/X_{^7\mathrm{Li}}$ calculated by Eq. (17) is small,

$$\nu_e\simeq(6.68\nu_{^1\mathrm{H},^1\mathrm{H}}+1.58\nu_{^3\mathrm{He},^4\mathrm{He}}+17.39\nu_{^7\mathrm{Li},^1\mathrm{H}})/25.65. \tag{22}$$

In the conditions cited above $\nu_{^3\mathrm{He},^4\mathrm{He}}\simeq 17$, $\nu_{^7\mathrm{Li},^1\mathrm{H}}\simeq 11$ and again $\nu_e=10$.

The relative importance of the ^{3}He-^{4}He link increases with model number. At the centre of model 4, that link dominates the ^{3}He-^{3}He reaction in the proportion 5.7/4 but at the point corresponding to the extremum of $\delta p/p$ in Fig. 1, both terminations have the same importance. In earlier models, the ^{3}He-^{3}He reaction strongly dominates in the vicinity of the extremum of $\delta p/p$.

The carbon cycle also contributes to the large value of ν_e; it yields a value of ν_e of 19.6 for it alone.

The values of the integral $E_N=\int(\delta T/T)\delta\varepsilon dm$ for the five models studied are given in Table 2. It should be noticed that E_N is always destabilizing and that the gradient of ^{3}He abundance appears explicitly nowhere in the lagrangian formalism; the effect of the ^{3}He-^{3}He reaction simply comes, as stated above, through the influence of $\nu_{^3\mathrm{He}}$ on ν_e.

The second term in expression (14) is written (Gabriel *et al.*, 1974)

$$\begin{aligned}
&\delta\left[\frac{1}{\varrho}\nabla\cdot(\boldsymbol{F}_R+\boldsymbol{F}_C)\right]=\frac{d\delta L_R^r}{dm}+\frac{d\delta L_c^r}{dm}-\frac{l(l+1)}{\varrho r^2}(\delta F_R^h+\delta F_c^h)\\
&-\frac{l(l+1)\chi}{\sigma^2 r^2}\frac{d(L_R+L_c)}{dm}+\frac{l(l+1)\chi}{\sigma^2 r^2}\cdot\frac{F_R^r+F_c^r}{\varrho r}
\end{aligned} \tag{23}$$

with

$$\frac{\delta L_k^r}{L_k}\equiv\frac{\delta F_k^r}{F_k^r}+2\frac{\delta r}{r} \quad (k=R,C), \tag{24}$$

$$\chi\equiv\frac{\delta p}{\varrho}+\delta\Phi. \tag{25}$$

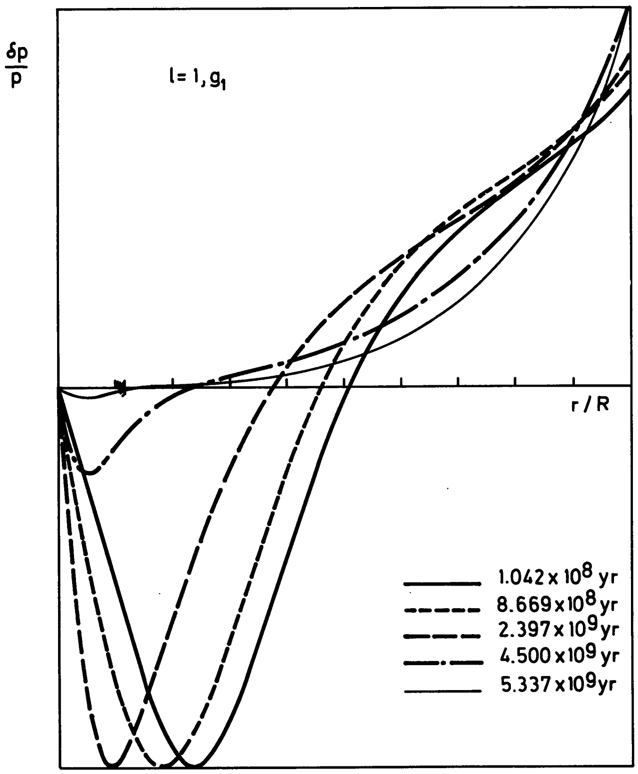

Fig. 1. Relative amplitude, $\delta p/p$, of the pressure variations for the mode $l=1, g_1$ in the five models. Models are identified by their age. Maximum scale for $r/R=1$. Value of $\delta p/p$ at $r/R=1$ is 5.54, 4.02, 2.82, 14.4 and 126 for models 1, 2, 3, 4, 5, respectively

Subscripts R and C mean respectively radiative and convective, while superscripts r and h refer to the radial and horizontal components of the perturbation of the vector fluxes, $\boldsymbol{F}_R$ and $\boldsymbol{F}_C$. Their contravariant components are δF^r, δF^θ, δF^φ, and $\delta F^r = \delta F^r(r) P_l^m \exp(\mathrm{i}m\varphi)$, $\delta F^h = r^2 \delta F^\theta / \{(\partial P_l^m/\partial\theta)\,[\exp(\mathrm{i}m\varphi)]\}$. One has, for the radiative flux

$$\frac{\delta F_R^r}{F_R^r} = \left[4 - \left(\frac{\partial \log \varkappa}{\partial \log T}\right)_\varrho\right]\frac{\delta T}{T} - \left[1 + \left(\frac{\partial \log \varkappa}{\partial \log \varrho}\right)_T\right]\frac{\delta \varrho}{\varrho} + \frac{d\left(\frac{\delta T}{T}\right)}{d\log T} - \frac{d\delta r}{dr}. \tag{26}$$

$\varkappa$ being the opacity coefficient,

$$\frac{\delta F_R^h}{F_R^r} = \frac{\delta T}{T}\left(\frac{d\log T}{dr}\right)^{-1} - \delta r + \frac{\chi}{\sigma^2 r}. \tag{27}$$

The effects of convection are estimated by a generalization of Unno's treatment (1967) for radial oscillations. Then (Gabriel et al., 1974)

$$\frac{\delta F_C^i}{F_C^r} = \left(\frac{\delta\varrho}{\varrho} + \frac{\delta T}{T}\right)\delta^{ir} + \frac{\delta \bar{V}^i}{\bar{V}^r} + \frac{\bar{V}^i \delta \Delta S}{V^r \Delta S} \quad (i=r,h) \tag{28}$$

where V^i is the i-component of the convective velocity, ΔS the excess entropy of the convective element and δ^{ir} the Kronecker symbol.

We then write in correspondence with each term in (23):

$$E_F = \int \frac{\delta T}{T}\,\delta\left[\frac{1}{\varrho}\nabla\cdot(\boldsymbol{F}_R + \boldsymbol{F}_C)\right] dm$$

$$= E_R^r + E_C^r + E_R^h + E_c^h + E_5 + E_6$$

Results for selected models are listed in Tables 2 and 3. E_F is always stabilizing although some partial contributions are not.

Let us, for discussion purposes, consider the g_1 mode for $l=1$. E_R^r is stabilizing in the Younger models and becomes destabilizing in the two last ones. In all models, however, the behaviour of the integrand $(\delta T/T)[d(\delta L_R^r)/dr]$ is qualitatively the same, the details depending on the values of the amplitudes.

Table 3. Detailed contributions to the vibrational stability coefficient (see text)

	Model number	E_R^r	E_C^r	E_R^h	E_C^h	E_5	E_6
$l=1\ g_1$	1	6.486(34)	1.665(34)	2.108(34)	−8.373(32)	6.029(34)	−2.695(34)
	2	2.769(34)	1.044(34)	1.338(34)	−5.941(32)	4.202(34)	−1.739(34)
	3	3.664(33)	6.591(33)	7.595(33)	−3.310(32)	2.539(34)	−9.343(33)
	4	−5.412(34)	1.784(35)	7.478(33)	−5.609(33)	2.269(34)	−8.416(34)
	5	−3.419(36)	1.908(37)	6.434(34)	−3.718(35)	1.879(34)	−3.043(34)
$l=1\ g_2$	1	1.305(35)	3.701(33)	2.328(34)	5.880(32)	5.099(34)	−3.148(33)
	2	2.416(34)	1.753(33)	8.353(33)	4.234(31)	1.778(34)	−3.510(33)
	3	4.151(33)	1.692(33)	3.808(33)	−6.345(31)	7.534(33)	−2.070(33)
	4	1.934(33)	3.747(33)	3.626(33)	−1.607(32)	6.378(33)	−1.983(33)
	5	−3.475(32)	1.089(34)	4.211(33)	−4.278(32)	7.114(33)	−2.216(33)
$l=1\ g_3$	1	7.982(34)	3.274(33)	1.132(34)	−4.395(32)	3.773(34)	−2.349(33)
	2	9.165(33)	2.243(32)	3.533(33)	3.075(31)	1.027(34)	−2.046(33)
	3	2.295(33)	4.415(32)	1.732(33)	2.649(30)	4.391(33)	−1.029(33)
	4	1.549(33)	9.707(32)	1.650(33)	−3.891(31)	3.477(33)	−8.686(32)
	5	1.769(33)	1.637(33)	1.924(33)	−7.061(31)	3.790(33)	−9.654(32)

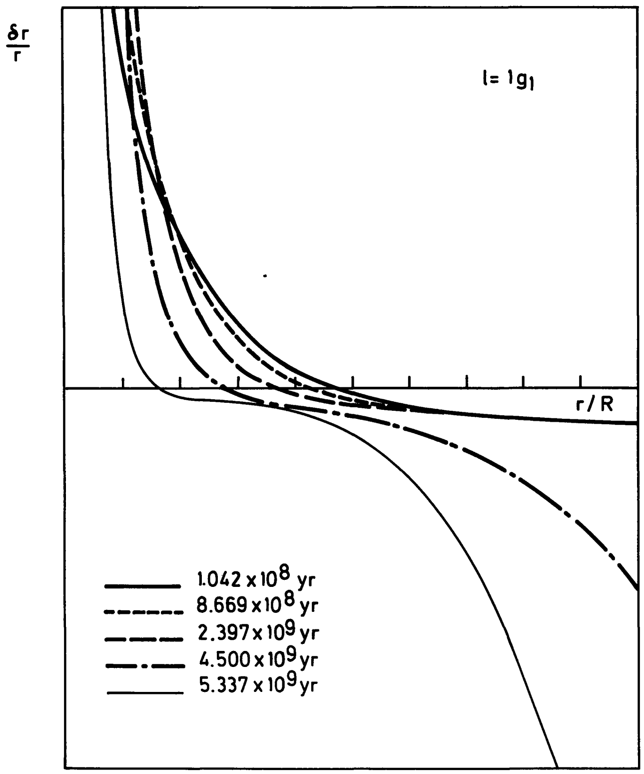

Fig. 2. Relative amplitude, $\delta r/r$, of radial distance variations for the mode $l=1$, g_1 in the five models. Value of $\delta r/r$ at $r/R=1$ is -1.19, -0.769, -0.419 and -1.32 for models 1, 2, 3 and 4 respectively; for model 5 value of $\delta r/r$ at $r/R=0.85$ is -5.19

Near the centre (Figs. 1 and 2), $\delta T/T$ is negative while $\delta r/r$ is positive and varies as r^{-1}. Moreover, $\delta L_R^r/L_R \simeq 2\delta r/r$ (since $\lim_{r\to 0}[(\delta T/T)/(r^2\delta r/r)] = \text{const.}$) so that $d(\delta L_R^r)/dr$ is positive and varies as r. Thus the integrand is negative in a region of mass fraction ~ 0.1 in model 1 and ~ 0.03 in model 4. Further out the usual stabilizing effect is restored until one reaches the convective envelope where the rapid fall of the radiative luminosity causes δL_R^r to decrease outwards, leading to destabilization as $\delta T/T$ is positive. The importance of the destabilizing effect of the convective zone increases with the amplitudes of $\delta T/T$ and $\delta p/p$ in that region: Fig. 1 shows that the weight of the convective layers is largest compared to the deeper damping layers in models 4 and 5, producing global destabilization. This behaviour shows that in such cases where the amplitudes are large in the convective envelope, neglecting the latter's contribution to E_R would amount to ignore part of the driving.

Consider now E_R^h and E_6 in the radiative part of the star. Taking Eq. (27) into account and grouping the third and the last terms of the right-hand side of Eq. (23) yields, for the integrand leading to the sum $E_R^h+E_6$, the expression $-l(l+1)T'/[T\varrho r^2(d\log T/dr)]$ where T' stands for the amplitude of the eulerian perturbation of temperature. Thus the layers where T' and δT have opposite signs are driving the pulsation. This can be intuitively understood with the asymptotic picture of an oscillating bubble. T' simply represents the temperature difference, at level $r+\delta r$, between a bubble originating from the level r and the surrounding material.

If $T'>0$ the bubble (in which, in an asymptotic mode, $\delta T<0$ as $\delta r>0$) radiates towards the surroundings and cools down.

Then its motion downward will have a larger amplitude than if it were adiabatic. There is in each model a destabilizing contribution to the sum $E_R^h+E_6$ coming from the central region within a radius fraction decreasing from 0.21 in model 1 to 0.13 in model 5. Then the integrand is positive (damping) in a large part of the star, up to radius fraction $\simeq 0.75$. Above, convection starts playing a rôle, adding a destabilizing contribution to E_6; there is no much change in E_R^h. The importance of the central destabilization is much reduced in model 5, due to the small relative amplitudes of $\delta T/T$ and T'/T while the stabilizing region harbours large amplitudes. That explains why the sum $E_R^h+E_6$ is positive in that model and negative (driving) in the others.

The radial convective term, E_c^r is stabilizing in all models. There are, however, in all models, zones driving and damping with respect to that term. In the deeper layers of the convective envelope, convection does not adapt itself to pulsation. Then (Gabriel *et al.*, 1974)

$$\frac{\delta L_c^r}{L_c} \simeq \frac{4}{3}\frac{\delta\varrho}{\varrho} + \frac{\delta T}{T} + 2\frac{\delta r}{r}, \tag{29}$$

δL_c^r increases outwards, which by (23) is a stabilizing effect as $\delta T/T$ is positive. As one comes progressively closer to the surface, the characteristic timescale of convection decreases so that convection follows more and more closely the pulsation.

In the case where convection is adiabatic and adapts instantaneously to pulsation, one has (Gabriel *et al.*, 1974)

$$\frac{\delta L_c^r}{L_c} \simeq \frac{\delta T}{T} + \frac{1}{2}\frac{\delta\varrho}{\varrho} + \frac{1}{2}\frac{\delta p}{p} - \frac{3}{2}\frac{d(\delta p/p)}{dr}\bigg/\frac{d\log p}{dr} - \frac{1}{2}\frac{\delta(C_p Q)}{C_p Q} \tag{30}$$

where C_P is the specific heat at constant pressure and $Q \equiv (\partial \log T/\partial \log \varrho)_p$. In the two limiting cases given by (29) and (30) (ignoring the last term), δL_c^r has a stabilizing effect.

However, Eq. (30) gives for $|\delta L_c^r/L_c^r|$ a smaller value than Eq. (29). That means that in the transition region where convection adapts itself partially to the pulsation, δL_c^r has a destabilizing influence. The term $\left[-\frac{1}{2}\frac{\delta(C_P Q)}{C_P Q}\right]$ in (30) has an effect similar to the γ-mechanism for radial oscillations but never invalidates the argument given above.

High enough, however, convection cannot be treated as adiabatic and the layers where the dissipation becomes appreciable are damping again.

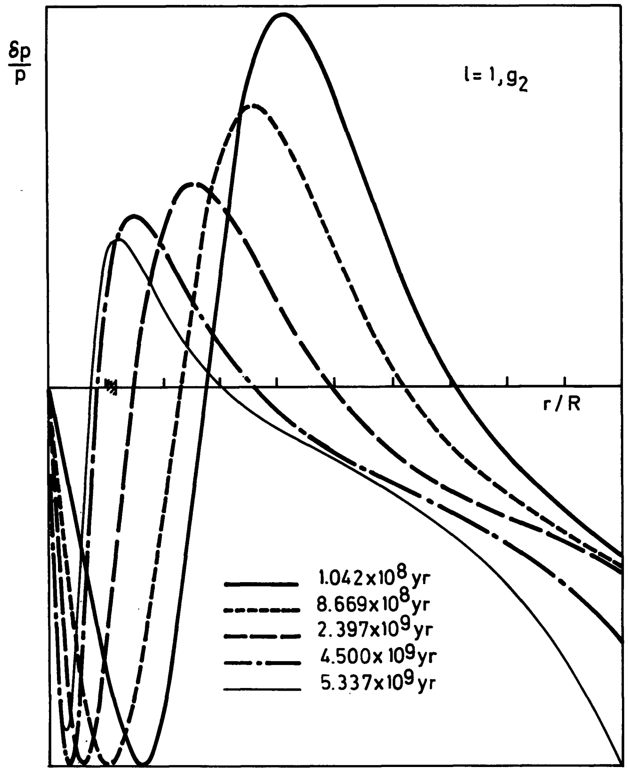

Fig. 3. Same as Fig. 1 for the mode $l=1$. Scale of $\delta p/p$ is arbitrary

In the competition between the nuclear term E_N and the flux term E_F, the importance of the nuclear driving strongly depends on the distribution of the amplitudes of $\delta p/p$ (or $\delta T/T$). Figure 1 shows that the node and the extremum of $\delta p/p$ come progressively toward the centre as the star evolves. Hatched lines indicate the layer where the energy generation is half its central value. As long as the node of $\delta p/p$ lies sufficiently outside that region, the evolution favours E_N since increasingly large values of $\delta T/T$ occur in the burning core. The situation is reversed as soon as the node comes close enough to the centre. The same behaviour takes place for the mode $l=1$, g_2 (Fig. 3). At given l, the higher the order of the mode, the closer to the centre the position of the node. This explains that all models are stable towards the mode $l=1$, g_3. Also, for a given mode, the ratio of the surface amplitudes to those in the core increases with l and all models are stable to modes $l=2$ and higher.

The second integral in Eq. (14) expresses the influence of the mechanical effects of convection: ε_2 represents the rate, per unit mass, of dissipation of turbulent kinetic energy into heat (Cowling, 1936; Ledoux and Walraven, 1958). $\bar{V}$ is the mean velocity of turbulence.

A few values of that integral are listed in Table 4 under the heading E_{ε_2}. It is always positive and produces driving, to such an extent that it would render all models unstable towards some modes. However the theory leading to that term is very uncertain. Therefore,

Table 4. Mechanical effects of convection (see text) ($l=1$, g_1)

Model number	E_{ε_2}
1	2.88(34)
2	8.82(33)
3	5.98(33)
4	2.51(35)
5	4.40(37)

E_{ε_2} has been neglected altogether in the calculation of the damping coefficient, whose inverse, σ'^{-1}, is tabulated in Table 2. A negative value of σ'^{-1} for a given mode, indicates that a model is unstable towards that mode.

For $l=1$, the star, as it evolves, becomes unstable to modes g_1 and g_2 at an age of about 2.4×10^8 years. Restabilization occurs at an age of about 3×10^9 years. The model representing the present sun is stable. All models are stable for modes g_3 and higher and for all modes corresponding to $l=2$.

One word must be added relative to the formalism used. As all models are very close to thermal equilibrium, σ' can be written indifferently in terms of eulerian or lagrangian variations. We used the latter but, for comparison purposes, σ' was also computed independently in the eulerian formalism. Deep enough in the star, results agree but, near the surface, the relative eulerian perturbations of ϱ and T increase indefinitely with $d\log\varrho/dr$. Div$\boldsymbol{F}'$ is the sum of terms proportional to these perturbations or their derivatives, which all separately increase. But div$\boldsymbol{F}'$, being equal to δdiv$\boldsymbol{F}$ in the outer layers, remains finite. This inevitably leads to a loss in numerical accuracy which, near the surface, becomes so poor that the eulerian results can no longer be trusted. Moreover, the physical interpretation is straightforward when one follows a given mass element. Thus, for easy interpretation as well as for numerical purposes, the lagrangian description, although algebraically slightly more cumbersome, is to be preferred.

V. Conclusions

After about 2.4×10^8 years on the main sequence, a star of solar mass and normal composition becomes unstable towards lower g modes of $l=1$ and becomes stable again after 3×10^9 years. In the linear approximation, there is of course no way of predicting the response of the star to such an instability, particularly regarding a possible mixing in part or all of the star. The growth time of the amplitude is never short compared to the Kelvin-Helmholtz time scale. This could favour a slow mixing rather than the instantaneous one proposed by Fowler (1972) to lower the solar neutrino flux. Note, however, that Ulrich and Rood

(1973) and Ulrich (1974) argue that the instability towards non-radial oscillations is not sufficient, by itself, to produce mixing and that it must be accompanied by another agent.

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A. Boury
M. Gabriel
A. Noels
R. Scuflaire
P. Ledoux
Institut d'Astrophysique
Université de Liège
B-4200 Cointe-Ougrée, Belgique

Erratum

There is a typo in equation (9), there is an extra ℓ factor in the first term of the second member. The equation should read

$$\frac{dv}{dx} = \frac{\ell+1}{x}(\ell u - v) + \frac{\ell(\ell+1)}{x\omega^2}\frac{4\pi R^3\rho}{M}\left(\frac{q}{x^3}y + \frac{RP}{GM\rho}z + u\right).$$