LINEAR RADIAL PERTURBATIONS OF SUPERMASSIVE STARS

R. SCUFLAIRE
Institut d'Astrophysique, Université de Liège, Cointe-Ougrée, Belgium

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Abstract. Numerical integration of the linear non-adiabatic equations for radial perturbations of physical models of supermassive stars confirm the existence of both dynamical and secular instabilities above a critical mass. The value of the latter agrees very well with that deduced from approximate stability criteria. The present analysis brings out for the first time the existence of a continuous transition between dynamical and secular modes in a sequence of models of different masses.

1. Introduction

Hoyle and Fowler (1963) have invoked the existence of supermassive stars with masses ranging from 10^5 to 10^8 $M_\odot$ to explain the fluxes from intense radiosources. The burning of hydrogen in a star of 10^8 $M_\odot$ could explain luminosities as high as 10^{45} or 10^{46} erg s^{-1} observed in the brightest sources (Matthews *et al.*, 1964). The discovery of quasars and the cosmological interpretation of their redshift (Matthews and Sandage, 1963; Schmidt, 1963; Greenstein and Matthews, 1963) seemed to confirm the point of view of Hoyle and Fowler. Other hypotheses have been put forward to explain the nature of quasars (see Demaret, 1969b, 1970). They may be divided into two classes. The former are variants of the hypothesis of the supermassive star, the latter interpret the quasar as a set of more or less independent objects (supernovae for instance or very massive stars). The statistical study of the fluctuations in luminosity seems to favour a unique object (Gudzenko *et al.*, 1968, 1971).

We shall not raise here the difficulties of formation of supermassive stars, we shall be mainly interested in the problem of their stability. As yet this problem has been investigated only with approximate stability criteria and the polytropic model of index three has been often used for this purpose. These criteria have shown that supermassive stars with masses greater than a critical mass (M_{cr}) of about 5×10^5 $M_\odot$ are dynamically and secularly unstable (see Section 3). In this paper we report the results of a study of the complete linear non-adiabatic radial perturbations of physical models of supermassive stars in the main sequence stage with masses in the neighbourhood of the critical mass.

2. Models

We have computed main sequence models of supermassive stars with masses ranging from 10^5 to 10^6 $M_\odot$. This domain of mass was chosen so as to include the critical mass

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of instability. We have supposed no rotation and no magnetic field. We have adopted for a chemical composition of population I the values of

$$X = 0.8, \qquad Y = 0.18, \qquad Z = 0.02,$$

$X_{\text{CNO}} = 1.2 \times 10^{-2}$ (Iben Mixture XVII of Cox and Stewart, 1970).

With the Schwarzschild's metric

$$ds^2 = e^{\nu} c^2\, dt^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{\lambda}\, dr^2,$$

the field equations of general relativity may be written

$$\frac{d\nu}{dr} = \frac{e^{\lambda} - 1}{r} + \frac{8\pi G}{c^4} r e^{\lambda} P, \tag{1}$$

$$\frac{d\lambda}{dr} = -\frac{e^{\lambda} - 1}{r} + \frac{8\pi G}{c^4} r e^{\lambda} \varepsilon, \tag{2}$$

$$\frac{dP}{dr} = -\tfrac{1}{2}(P + \varepsilon)\frac{d\nu}{dr}, \tag{3}$$

where ε is the energy density, including the rest mass energy and the internal energy

$$\varepsilon = \varrho c^2 + \varrho U.$$

The equations governing the hydrostatic structure may be written in a form similar to the classic one (Oppenheimer and Volkoff, 1939). With the mass m up to the value r of the radius defined by

$$e^{-\lambda} = 1 - \frac{2Gm}{rc^2},$$

we get

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon,$$

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\frac{P + \varepsilon}{c^2}\left(1 - \frac{2Gm}{rc^2}\right)^{-1}\left(1 + \frac{4\pi r^3 P}{mc^2}\right).$$

As far as the equations of thermal equilibrium and transfer are concerned, since the uncertainties on the rate of nuclear energy generation ε_N are greater than the general relativistic corrections we keep their classical forms

$$\frac{dL}{dr} = 4\pi r^2 \varrho \varepsilon_N,$$

$$\frac{dT}{dr} = -\frac{3\kappa\varrho L}{16\pi r^2 acT^3}, \quad \text{(radiative zone)} \tag{4}$$

$$\frac{dT}{dr} = \left(\frac{dT}{dr}\right)_c. \quad \text{(convective zone)}$$

The second member of the latter equation is computed according to the usual mixing length theory of convection (Vitense, 1953; Böhm-Vitense, 1958) with a mixing length equal to the pressure scale height. In our models, the convection is very efficient and to a very good approximation we can write for the convective flux

$$F_c = \frac{1}{4\sqrt{2}} C_P \varrho T \frac{QP}{\varrho} (\nabla - \nabla_{ad})^{3/2}, \tag{5}$$

where

$$Q = -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_P, \qquad \nabla = \frac{\mathrm{d} \ln T}{\mathrm{d} \ln P}, \qquad \nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_S.$$

In supermassive stars density is low and temperature is high. The matter is completely ionized and the equation of state is that of a mixture of an ideal monoatomic gas and radiation. The pressure of the gas represents only a slight fraction of the total pressure i.e.,

$$\beta = \frac{P_G}{P} \ll 1.$$

Thus we have

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \varrho}\right)_S = \frac{4}{3} + \frac{1}{6}\beta + O(\beta^2) \simeq \frac{4}{3}.$$

The opacity is essentially due to electronic scattering. It is a constant given by

$$\kappa = 0.2(1 + X).$$

Nuclear energy is produced by the carbon cycle. In the domain of temperature where it works in our models, the rate of production is approximately given by

$$\varepsilon_N = 1.456 \times 10^{-13} \varrho\, X X_{\mathrm{CNO}}\, T_6^{12.7}.$$

The models have been described in details and discussed elsewhere (Scuflaire, 1975a, b). They are composed of an extended convective core (0.9 of the radius) and a radiative envelope containing a slight fraction of the mass (of the order of 2×10^{-4}). The structure is close to that of a classical polytrope of index 3. The luminosity is proportional to the mass so that the lifetime of supermassive stars on the main sequence is independent of the mass and is of the order of 2×10^6 yr. Table I gives

TABLE I

Some characteristics of the models

$10^{-5} \frac{M}{M_\odot}$	ϱ_c (g cm^{-3})	T_c (K)	β_c	R (cm)	L (erg s^{-1})	T_e (K)	E (erg)	$2\frac{GM}{Rc^2}$
1	7.25(−2)	5.59(7)	2.33(−2)	3.21(13)	1.35(43)	6.55(4)	−1.20(54)	9.17(−4)
2	5.46(−2)	5.72(7)	1.65(−2)	4.48(13)	2.72(43)	6.61(4)	−2.04(54)	1.32(−3)
3	4.63(−2)	5.79(7)	1.35(−2)	5.44(13)	4.10(43)	6.64(4)	−2.47(54)	1.63(−3)
4	4.12(−2)	5.85(7)	1.17(−2)	6.24(13)	5.48(43)	6.67(4)	−2.46(54)	1.89(−3)
5	3.76(−2)	5.89(7)	1.05(−2)	6.94(13)	6.86(43)	6.69(4)	−2.02(54)	2.13(−3)
7	3.28(−2)	5.96(7)	8.84(−3)	8.14(13)	9.63(43)	6.72(4)	0.23(54)	2.53(−3)
10	2.83(−2)	6.03(7)	7.39(−3)	9.66(13)	1.38(44)	6.75(4)	7.06(54)	3.05(−3)

some characteristics of the models at the centre (ϱ_c, T_c, β_c) and at the surface (R, L). E is the total energy. The parameter $2GM/Rc^2$ gives the importance of relativistic effects which remain small in the domain of mass considered as far as the structure of the star is concerned.

3. Stability Criteria

Iben (1963) has shown that relativistic polytropes with sufficiently large masses have a positive total energy and are consequently dynamically unstable. Chandrasekhar (1964) has derived a variational principle giving the squares of the frequencies of the radial normal modes as the stationary values of an integral expression. Applied to a polytropic model of index 3 and constant adiabatic coefficient Γ_1, this principle provides the following criterion of stability

$$\Gamma_1 > \frac{4}{3} + 1.1245\,\frac{2GM}{Rc^2}. \tag{6}$$

This criterion differs from the classical one by the presence of the term in $2GM/Rc^2$. A criterion based on the consideration of the energy as a function of radius in a sequence of homologous models has been used by Fowler (1964, 1966a) and proved by Thorne (1966a Section 4.1.4, 1966b Section 4.2.4). This energy method results in the same criterion (6). The same condition of stability is also obtained from the expression of σ^2 (σ being the angular frequency) of an adiabatic homologous pulsation derived by Fowler (1966b) using the virial method. Though these criteria cannot apply exactly to our physical models we used them and found a critical mass of about $3.7\times10^5 M_\odot$.

Appenzeller and Kippenhahn (1971) have shown that supermassive stars with sufficiently large masses are secularly unstable. The criterion used by these authors is derived from the consideration of homologous transformations and neglects the transfer of energy during the perturbation. According to Osaki (1972) this instability is not distinct from the dynamical instability. Demaret and Ledoux (1973) have rediscussed the problem and shown that secular instability appears simultaneously with and as a consequence of dynamical instability.

4. Computation of Linear Radial Perturbations

As the non-perturbed model is a static one, every perturbation can be described as a superposition of normal modes which depend on time as e^{st} where s is generally a complex coefficient. The computation of normal modes of linear radial perturbation leads to the resolution of a boundary-value problem described below.

Although relativistic corrections have only a small effect on the structure of the star, this is not true of its stability. Indeed it is the relativistic terms in the dynamical equations which introduce the possibility of dynamical instability. Let δf and f' be respectively the lagrangian and eulerian perturbations of any quantity f. With these notations Equations (39), (43) and (51) of Chandrasekhar (1964) may be written as

$$\varepsilon' = -\delta r \frac{\mathrm{d}\varepsilon}{\mathrm{d}r} - (P + \varepsilon)\frac{e^{\nu/2}}{r^2}(r^2 e^{-\nu/2}\,\delta r)$$

$$- s^2 e^{\lambda-\nu}\frac{P+\varepsilon}{c^2}\delta r = \frac{\mathrm{d}P'}{\mathrm{d}r} + P'\frac{\mathrm{d}}{\mathrm{d}r}(\tfrac{1}{2}\lambda + \nu) + + \tfrac{1}{2}\varepsilon'\frac{\mathrm{d}\nu}{\mathrm{d}r} - \tfrac{1}{2}(P+\varepsilon)\left(\frac{\mathrm{d}\nu}{\mathrm{d}r} + \frac{1}{r}\right)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}r} + \frac{\mathrm{d}\nu}{\mathrm{d}r}\right)\delta r, \tag{7}$$

$$n' = -\delta r\frac{\mathrm{d}r}{\mathrm{d}n} - n\frac{e^{\nu/2}}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 e^{-\nu/2}\,\delta r); \tag{8}$$

n being the number of baryons per unit volume. The latter equation may be written as

$$\frac{\delta\varrho}{\varrho} = -\frac{e^{\nu/2}}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 e^{-\nu/2}\,\delta r). \tag{9}$$

From Equations (8) and (28) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta r}{r}\right) = \left(-\frac{3}{r} + \frac{1}{P+\varepsilon}\frac{\mathrm{d}P}{\mathrm{d}r}\right)\frac{\delta r}{r} - \frac{1}{r}\frac{\delta\varrho}{\varrho}. \tag{10}$$

With the aid of Equation (7) ε' is eliminated from (8). Taking into account Equations (1), (2) and (3) we obtain, after some tedious algebra,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta P}{P}\right) = \left\{-\frac{4}{P}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{8\pi G}{c^4}re^{\lambda}(P+\varepsilon) + \frac{r}{P(P+\varepsilon)}\left(\frac{\mathrm{d}P}{\mathrm{d}r}\right)^2 - - \frac{r(P+\varepsilon)}{c^2 P}e^{\lambda-\nu}s^2\right\}\frac{\delta r}{r} + \left\{-\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{1}{P+\varepsilon}\frac{\mathrm{d}P}{\mathrm{d}r} - - \frac{8\pi G}{c^4}re^{\lambda}(P+\varepsilon)\right\}\frac{\delta P}{P}. \tag{11}$$

At the classical limit ($c \to \infty$) Equations (10) and (11) reduce to the usual equations of continuity and movement respectively (Ledoux and Walraven, 1958; Ledoux, 1963, 1969). Let us note that the energy-momentum tensor used by Chandrasekhar (1964) does not contain any term of transfer (see Demaret, 1969a). Thus, relativistic terms arising from the energy transfer have been omitted from the dynamical equations (10) and (11). This omission is consistent with our approximation of neglecting relativistic terms in the equations of conservation and transfer of energy. The conservation of thermal energy yields

$$\varrho T\frac{\mathrm{d}S}{\mathrm{d}t} = \varrho\varepsilon_N - \frac{1}{4\pi r^2}\frac{\partial L}{\partial r};$$

its linearized form being

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta L}{L}\right) = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}r}\left(\frac{\delta\varepsilon_N}{\varepsilon_N} - \frac{\delta L}{L}\right) - \frac{4\pi r^2 \varrho}{L} sT\,\delta S. \tag{12}$$

In a radiative zone, the linearized form of the transfer Equation (4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta T}{T}\right) = \frac{\mathrm{d}\ln T}{\mathrm{d}r}\left\{\frac{\delta L}{L} - 4\frac{\delta T}{T} - 4\frac{\delta r}{r} + \frac{\delta\kappa}{\kappa}\right\}. \tag{13}$$

In a convective zone the treatment of the transfer is more complex and we postpone its discussion to the next section.

At the centre regularity conditions must be satisfied, which can be written as

$$3\frac{\delta r}{r} + \frac{\delta\varrho}{\varrho} = 0, \tag{14}$$

$$sT\,\delta S + \varepsilon_N\left(\frac{\delta L}{L} - \frac{\delta\varepsilon_N}{\varepsilon_N}\right) = 0. \tag{15}$$

At the surface we have imposed two conditions which arise from the following considerations. The radiative relaxation time in the external layers of the star is much shorter than the characteristic times of the modes we shall study so that at each time the atmosphere of the star is in radiative equilibrium. This conditions may be expressed (cf. Baker and Kippenhahn, 1965) by

$$\frac{\delta L}{L} - 4\frac{\delta T}{T} - 2\frac{\delta r}{r} + \frac{\tau}{\tau + \frac{2}{3}}\left(\frac{\delta\kappa}{\kappa} - 2\frac{\delta r}{r}\right) = 0, \tag{16}$$

where τ is the optical depth of the point at which this condition is applied. The usual dynamical condition has been modified in order to take into account the large radiation pressure. It becomes

$$(\tau + \tfrac{2}{3}u)\frac{\delta P}{P} + \left(4\tau - \frac{r^3 s^2}{GM}\tau + \tfrac{4}{3}u\right)\frac{\delta r}{r} - \tfrac{2}{3}u\frac{\delta L}{L} = 0, \tag{17}$$

where

$$u = \frac{\kappa L}{4\pi c G M}.$$

When the radiation pressure is weak, $u \ll 1$ and (39) reduces to the usual dynamical condition

$$\frac{\delta P}{P} + \left(4 - \frac{r^3 s^2}{GM}\right)\frac{\delta r}{r} = 0.$$

In the case of supermassive stars u is very close to unity and Equation (17) must be used.

The perturbations of thermodynamic quantities may be expressed in terms of the perturbations of two of them. We have the relations

$$\frac{\delta P}{P} = \Gamma_1\frac{\delta\varrho}{\varrho} + \frac{(\Gamma_3 - 1)c_v \varrho T}{P}\frac{\delta S}{c_v}, \tag{18}$$

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta\varrho}{\varrho} + \frac{\delta S}{c_v}. \tag{19}$$

The perturbations of opacity vanish since opacity is constant: i.e.,

$$\frac{\delta\kappa}{\kappa} = 0.$$

The perturbations of the rate of energy production can be written as

$$\frac{\delta\varepsilon_N}{\varepsilon_N} = \frac{\delta\varrho}{\varrho} + \nu(s)\frac{\delta T}{T},$$

where the sensitivity to temperature $\nu(s)$ is computed from the linearized form of the kinetic equations for the nuclear reactions.

The problem to be solved is a linear non self-adjoint boundary-value problem described by the differential Equations (10), (11), (12), (13) and the boundary conditions (14), (15), (16) and (17). It has been numerically solved in the following manner. The differential equations were replaced by difference equations. For a given value of s, let $D(s)$ be the determinant of this linear system. The eigenvalues of the problem are those values of s for which $D(s)$ vanishes. They were obtained by successive iterations, using the secant method, from

$$s_{i+1} = \frac{s_i D(s_{i-1}) - s_{i-1} D(s_i)}{D(s_{i-1}) - D(s_i)}.$$

5. Treatment of Convection

The convection flux may be written as

$$F_c = \varrho T \langle V\,\Delta S\rangle,$$

where V is the radial component of the velocity of a turbulent element relative to the ambient medium and ΔS is its deviation of entropy per gram compared to the mean value in the medium. If we take the pressure scale height as the mixing length, the usual theory in a static situation gives the convective flux F_c as a function F_{c0} of ϱ, T and the gradients of P and T. Expression (5) is of this form. The static perturbation of such an expression is of the form

$$\left(\frac{\delta F_c}{F_c}\right)_0 = \frac{\partial \ln F_{c0}}{\partial \ln \varrho}\frac{\delta\varrho}{\varrho} + \frac{\partial \ln F_{c0}}{\partial \ln T}\frac{\delta T}{T} + \\ + \frac{\partial \ln F_{c0}}{\partial\left(\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right)}\frac{\delta\left(\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right)}{\frac{\mathrm{d}\ln P}{\mathrm{d}r}} + \frac{\partial \ln F_{c0}}{\partial\left(\frac{\mathrm{d}\ln T}{\mathrm{d}r}\right)}\frac{\delta\left(\frac{\mathrm{d}\ln T}{\mathrm{d}r}\right)}{\frac{\mathrm{d}\ln T}{\mathrm{d}r}}. \tag{20}$$

When the physical quantities describing the average medium vary in time, expression (20) remains valid only if the characteristic time of these variations are much longer than the characteristic time of convection, i.e. in the limit case $s \to 0$. In the general case, we may write

$$\frac{\delta F_c}{F_c} = \frac{\delta \varrho}{\varrho} + \frac{\delta T}{T} + \frac{\delta \langle V \Delta S \rangle}{\langle V \Delta S \rangle}.$$

The problem consist in evaluating the last term in the second member. Since convection is very effective in our models, we may think that the entropy of the convective element is not affected by the perturbation. It is more difficult to estimate the effects of compression or dilatation upon the radial component of the turbulent velocity V. In the limit case $s \to \infty$ (i.e. when the perturbation is faster than convection) we shall admit that V varies in the inverse ratio of the radial dilatation (Batchelor, 1955). Let us note that in the case of isotropic turbulence this assumption is consistent with Equation (53.28) of Ledoux and Walraven (1958) giving the fluctuation of turbulent pressure. Thus we have

$$\frac{\delta V}{V} = -\frac{\mathrm{d}\,\delta r}{\mathrm{d}r},$$

$$\frac{\delta \langle V \Delta S \rangle}{\langle V \Delta S \rangle} = -\frac{\mathrm{d}\,\delta r}{\mathrm{d}r},$$

and

$$\left(\frac{\delta F_c}{F_c}\right)_\infty = \frac{\delta \varrho}{\varrho} + \frac{\delta T}{T} - \frac{\mathrm{d}\,\delta r}{\mathrm{d}r}.$$

The subscript ∞ means that this expression is valid only in the case $s \to \infty$. We have obtained the same expression from the discussion of evolution equations established for the convective elements (Scuflaire, 1975a). In intermediate cases where the characteristic times of convection and perturbation are of the same order of magnitude we shall use an interpolation formula inspired from the treatment of convection of Cox (1967). According to him, we should have

$$\frac{\delta F_c}{F_c} = \frac{1}{1 + s\tau}\left(\frac{\delta F_c}{F_c}\right)_0, \tag{21}$$

where τ is the characteristic time of convection. This equation implies that

$$\left(\frac{\delta F_c}{F_c}\right)_\infty = 0.$$

In order to take into account a non-vanishing $(\delta F_c/F_c)_\infty$ we shall write

$$\frac{\delta F_c}{F_c} = \frac{1}{1 + s\tau}\left(\frac{\delta F_c}{F_c}\right)_0 + \frac{s\tau}{1 + s\tau}\left(\frac{\delta F_c}{F_c}\right)_\infty. \tag{22}$$

The results described in the following sections have been obtained using expression

(22). Nevertheless we have also carried out computations with expression (21) and we have found relative deviations of only 10^{-4} for the characteristic values.

6. Quasi-adiabatic Approximation (Dynamical Modes)

In the adiabatic approximation thermal phenomena are neglected. In Equations (18) and (19) one puts

$$\delta S = 0;$$

and the linearized equations of state reduce to

$$\frac{\delta P}{P} = \Gamma_1 \frac{\delta \varrho}{\varrho}, \qquad \frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta \varrho}{\varrho}.$$

One keeps only the dynamical Equations (10) and (11). The normal modes we get in this approximation are so-called dynamical modes. The problem is of the second order and may be put in a self-adjoint form with s^2 as an eigenvalue (Chandrasekhar, 1964). An eigenvalue s^2 is associated with two normal modes with opposite characteristic values s. If an eigenvalue s^2 is positive there exists a normal mode whose amplitude grows exponentially with time. This mode is said to be dynamically unstable. When all eigenvalues s^2 are negative all the normal modes consist in sinusoidal oscillations and the model is dynamically stable. One can write

$$s = \pm i\sigma,$$

where σ is the frequency of the oscillation.

Due to the non-adiabatic terms neglected in this approximation, the amplitude of the oscillation – in the case of a stable mode – will slowly vary with the time. According as this amplitude is increasing or decreasing the mode is said to be vibrationally unstable or stable. One obtains an approximation of the real part s_1 of s with the expression

$$s_1 = \frac{1}{2\sigma^2}\,\frac{\displaystyle\int \frac{\delta T}{T}\left(\delta\varepsilon_N - \frac{\mathrm{d}\,\delta L}{\mathrm{d}m}\right)\mathrm{d}m}{\displaystyle\int |\delta r|^2}. \tag{23}$$

We call this approximation quasi-adiabatic. It is not necessary to use the relativistic form of (23). However, it is important to limit the domain of integration. It seems that better values of s_1 are obtained if the upper limit of integration is taken at the point where the adiabatic approximation ceases to be valid. This point is defined (cf. Ledoux, 1969) by

$$\left|\frac{\delta T}{T}\right| = \frac{1}{\sigma c_v T}\left|\frac{\mathrm{d}\,\delta L}{\mathrm{d}m}\right|,$$

where δL is computed from the transfer equation with the adiabatic solution.

We have computed the fundamental dynamical modes $D_0^{\pm}$ and the first overtones $D_n^{\pm}$. The fundamental modes are dynamically stable when the mass of the star is less than $3.701 \times 10^5\ M_{\odot}$. Above this value the mode D_0^+ is dynamically unstable. This critical mass agrees with that deduced from criterion (6). Figure 1 shows that the fundamental mode is almost homologous. This gives sufficient weight to nuclear reactions to render vibrationally unstable the models with mass less than the critical mass (Demaret, 1972). Table II gives the value of s for the fundamental dynamical modes for different value of the mass. In the neighbourhood of the critical mass, s is

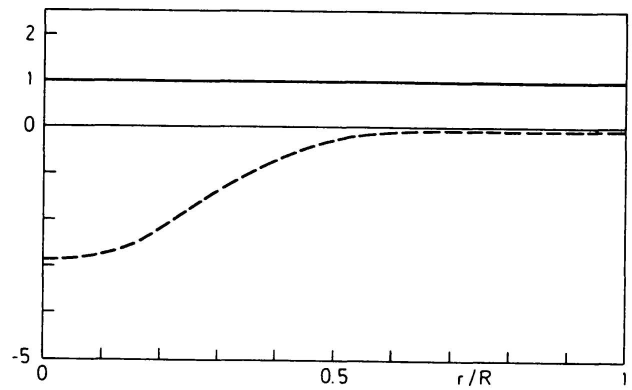

Fig. 1. Model of $10^5\ M_{\odot}$; characteristic functions $\delta r/r$ (——) and $\delta L/L$ (----) of modes $D_0^{\pm}$ in the quasi-adiabatic approximation.

TABLE II

Characteristic values of modes D_0^+ and D_0^- in the quasi-adiabatic approximation

$10^{-5}\frac{M}{M_{\odot}}$	s (s^{-1})
1	$9.827(-11) \pm 6.603(-6)i$
2	$1.532(-10) \pm 3.832(-6)i$
3	$3.670(-10) \pm 2.050(-6)i$
3.5	$1.274(-9) \pm 1.024(-6)i$
3.68	$1.220(-8) \pm 3.234(-7)i$
3.69	$2.317(-8) \pm 2.343(-7)i$
3.70	$2.503(-7) \pm 7.120(-8)i$
3.71	$\pm 2.110(-7)$
3.72	$\pm 3.070(-7)$
4	$\pm 1.117(-6)$
5	$\pm 2.221(-6)$
7	$\pm 3.046(-6)$
10	$\pm 3.591(-6)$

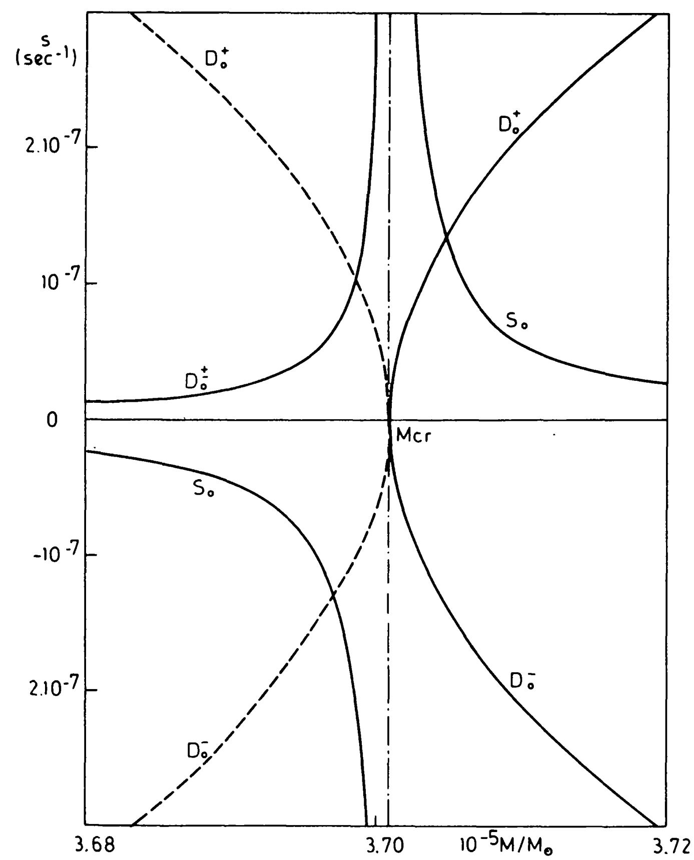

Fig. 2. Characteristic value s as a function of mass for modes $D_0^{\pm}$ (quasi-adiabatic approximation) and S_0 (quasi-static approximation). Real part: ——; imaginary part: ---.

TABLE III

Values of ω^2 for the dynamical harmonics for the classical polytrope of index 3 and $\Gamma_1 = 4/3$ and for the model of $10^5\ M_\odot$

Mode	Polytrope	$10^5\ M_\odot$
D_1	9.009	8.977
D_2	1.837(1)	1.834(1)
D_3	3.032(1)	3.026(1)
D_4	4.484(1)	4.472(1)
D_5	6.194(1)	6.167(1)
D_6	8.158(1)	8.117(1)
D_7	1.038(2)	1.030(2)
D_8	1.284(2)	1.273(2)
D_9	1.556(2)	1.537(2)
D_{10}	1.853(2)	1.820(2)

TABLE IV

Characteristic values of the dynamical harmonics in the quasi-adiabatic approximation for the model of $10^5\ M_\odot$

Mode	s (s^{-1})
D_1	$-3.531(-10) \pm 5.896(-5)i$
D_2	$-6.305(-9) \pm 8.556(-5)i$
D_3	$-4.678(-8) \pm 1.099(-4)i$
D_4	$-2.103(-7) \pm 1.336(-4)i$
D_5	$-6.652(-7) \pm 1.569(-4)i$
D_6	$-1.624(-6) \pm 1.800(-4)i$
D_7	$-3.115(-6) \pm 2.028(-4)i$
D_8	$-4.994(-6) \pm 2.254(-4)i$
D_9	$-6.659(-6) \pm 2.477(-4)i$
D_{10}	$-6.427(-6) \pm 2.695(-4)i$

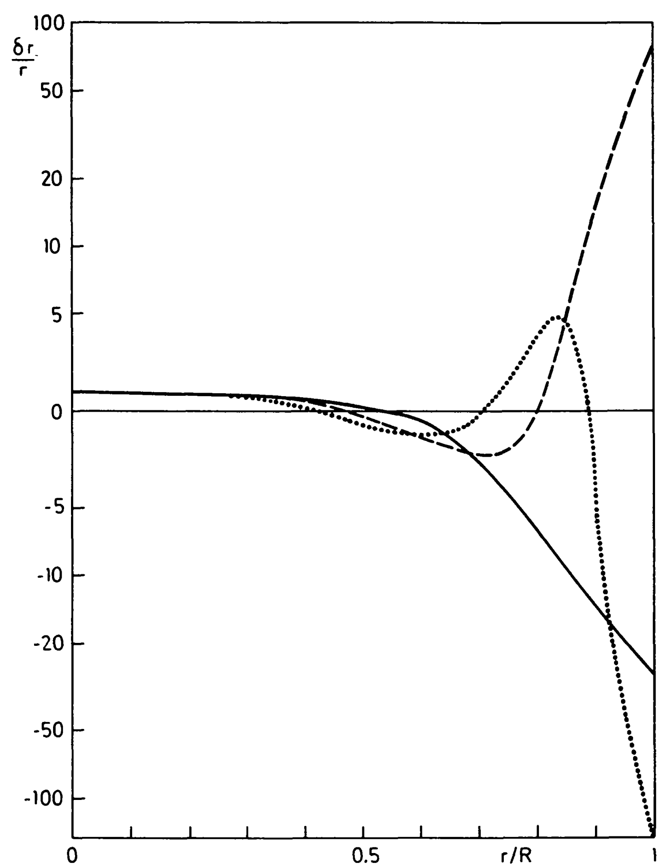

Fig. 3. Model of $10^5\ M_\odot$; characteristic functions $\delta r/r$ of modes $D_1^\pm$ (——), $D_2^\pm$ (---) and $D_3^\pm$ (· · ·).

represented as a function of the mass in Figure 2. Let us note that the real part of s tends to infinity as the critical mass is approached from below. This is due to the presence of σ^2 in the denominator of (23). This indicates that the adiabatic approximation is no longer valid close to the critical mass.

With regard to the overtones we have found that the corresponding modes of the classical polytrope of index 3 with $\Gamma_1 = \frac{4}{3}$ are good approximations. Table III compares the values of ω^2 for the polytropic model and a supermassive star model. ω is a dimensionless frequency defined as

$$\omega^2 = \frac{R^3\sigma^2}{GM}.$$

The overtones have been found vibrationally stable (Table IV). The damping is due to the heat transfer and increases rapidly with the order of the mode. Figure 3 shows the characteristic function $\delta r/r$ for the first three harmonics.

7. Quasi-static Approximation (Secular Modes)

The heat transfer and the generation of nuclear energy play an important part in the determination of secular modes. The dynamical terms are generally negligible, that is the term in s^2 is omitted from Equations (11) and (17). Secular modes are computed as if the model would remain in hydrostatic equilibrium during the perturbation. Thus we shall refer to this approximation as the quasi-static approximation. It is justified as long as characteristic secular time is much longer than characteristic dynamical time.

Our computations confirm the existence of a secular instability for the higher values of the mass (Appenzeller and Kippenhahn, 1971). The critical mass for the instability

TABLE V

Characteristic values of mode S_0 in the quasi-static approximation

$10^{-5}\frac{M}{M_\odot}$	s (s^{-1})
1	−2.070(−10)
2	−3.167(−10)
3	−7.951(−10)
3.5	−2.497(−9)
3.68	−2.410(−8)
3.69	−4.602(−8)
3.70	−4.991(−7)
3.71	5.663(−8)
3.72	2.666(−8)
4	1.703(−9)
5	3.918(−10)
7	1.523(−10)
10	7.837(−11)

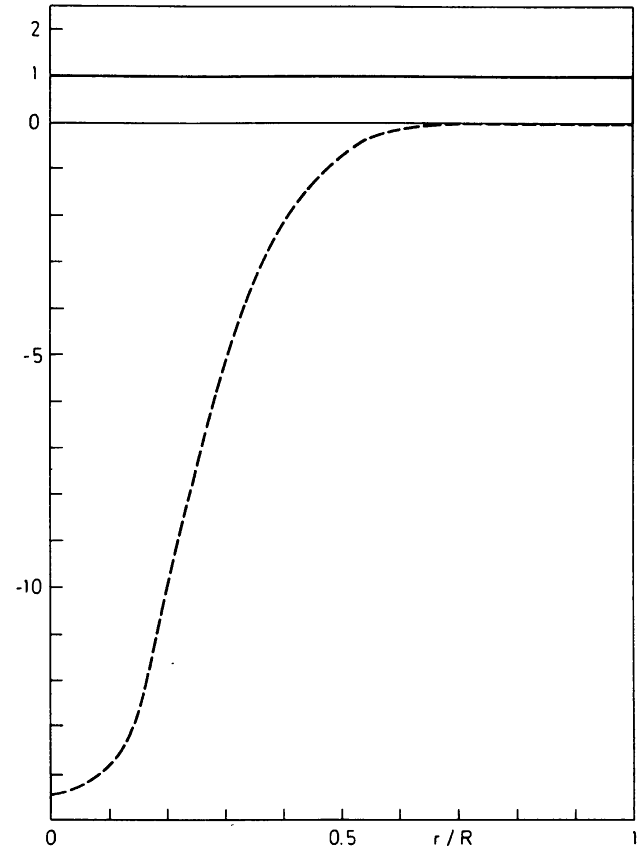

Fig. 4. Model of $10^5\ M_\odot$; characteristic functions $\delta r/r$ (——) and $\delta L/L$ (---) of mode S_0 in the quasi-static approximation.

coincides with the critical mass for dynamical instability. This fact corroborates the point of view of Demaret and Ledoux (1973) according to which both instabilities, though distinct, appear simultaneously. Table V gives the characteristic values associated to the secular fundamental mode S_0 for a few values of the mass. In the neighbourhood of the critical mass, the variations of $s(S_0)$ with the mass are represented in Figure 2. Let us note that s is discontinuous at the critical mass and tends to infinity. This is incompatible with the quasi-static approximation and indicates that close to the critical mass this approximation must be abandoned. Figure 4 shows that the mode S_0 is homologous. It differs from the modes D_0^+ and D_0^- by the values of $\delta L/L$. The characteristic value of the fundamental secular mode is linked with the Helmholtz-Kelvin time. Let the latter be defined as

$$\tau_{HK} = \frac{1}{L}\frac{\mathrm{d}E}{\mathrm{d}\ln R},$$

where $E(R)$ is the total energy of a hydrostatic model of radius R homologous to the considered one. With this definition we have noted the very simple relation

$$s\tau_{HK} \simeq -16.$$

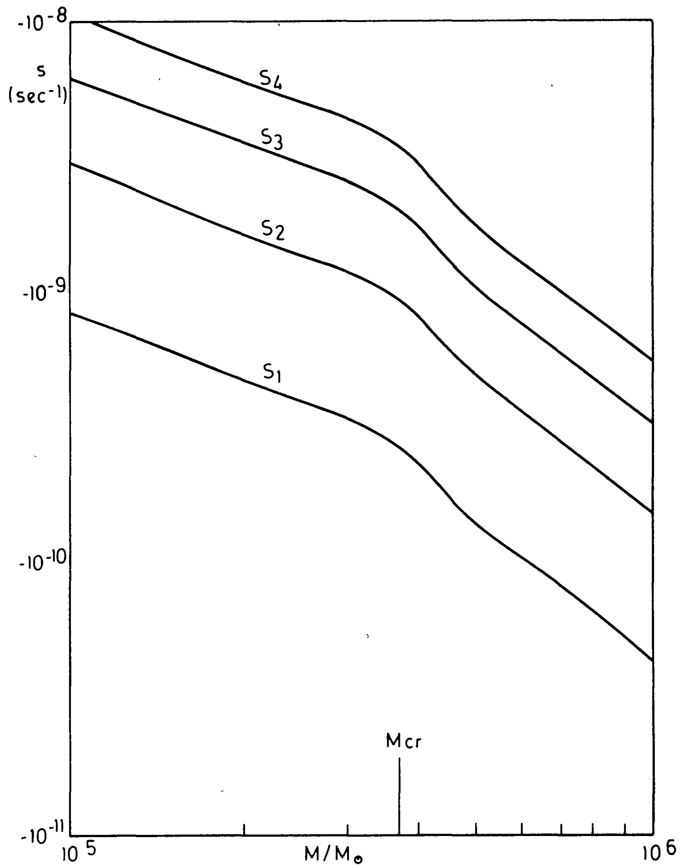

Fig. 5. Characteristic values s as a function of mass for modes S_1, S_2, S_3, S_4.

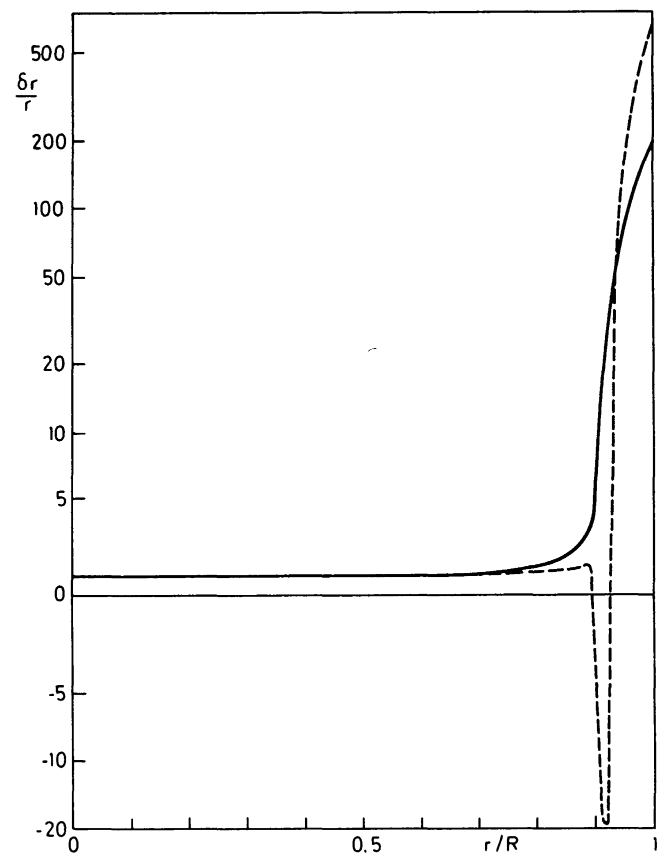

Fig. 6. Model of 10^5 $M_\odot$; characteristic functions $\delta r/r$ for modes S_1 (——) and S_2 (---).

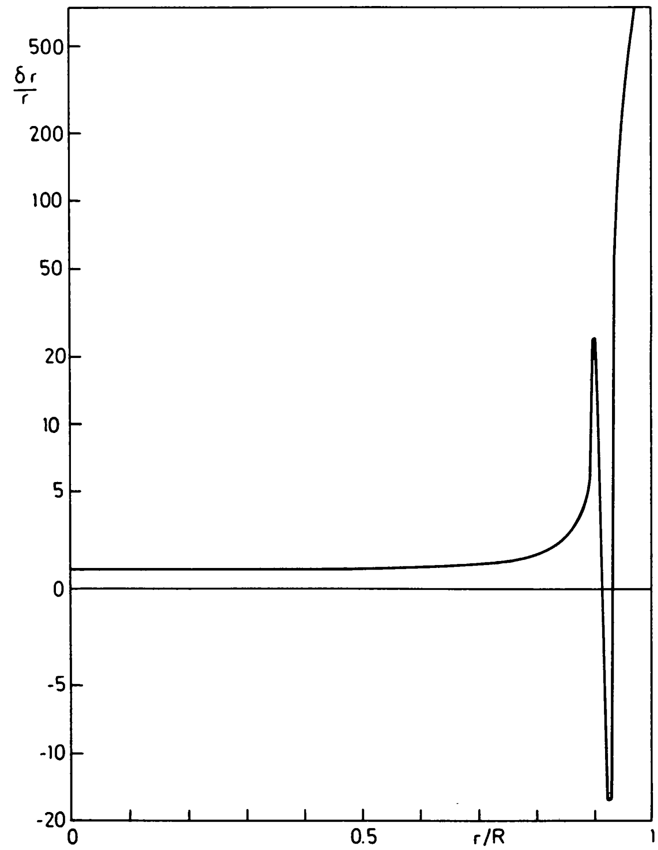

Fig. 7. Model of 10^5 $M_\odot$; characteristic function $\delta r/r$ for mode S_3.

The secular harmonics S_n have been found stable for all masses. Figure 5 represents the characteristic values of the first four harmonics as a function of mass. It must be noted that the following ratios are independent of mass

$$s(S_2)/s(S_1) = 3.48,$$
$$s(S_3)/s(S_1) = 7.34,$$
$$s(S_4)/s(S_1) = 12.4,$$

Figures 6 and 7 show the characteristic functions of the first three harmonics. It must be noticed that the perturbation is homologous in the convective core (0.9 of the radius). This can be explained by the following considerations. The characteristic times of these modes are longer than the convection time which is of the order of 10^7 s. Thus, during the perturbation, the convective core remains in adiabatic equilibrium – i.e., keeps a polytropic structure of index 3, homologous to its unperturbed structure.

8. Complete Treatment

As the quasi-adiabatic and quasi-static approximation are no longer valid close to the critical mass (Sections 6 and 7) it is then necessary to keep all the terms in the perturbation equations. The linear non-adiabatic radial problem has been treated recently by several authors (Castor, 1971; Ziebarth, 1970; Iben, 1971; Davey, 1973; Percy, 1975). We have developed our own program for the complete treatment of radial perturbations of stars and applied it to the computation of both dynamical and secular modes.

Far from the critical mass, the three fundamental modes obtained with the complete treatment, which we call F_1, F_2, F_3, may be identified exactly with the fundamental modes obtained in the approximate treatment D_0^+, D_0^-, S_0 (compare Table VI with Tables II and V). But in the neighbourhood of the critical mass, such an identification is no longer possible. As it is shown in Figure 8 the three characteristic values contrarily to the previous approximations are continuous functions of the mass. The identifications with modes obtained in the preceding sections differ on both sides of the critical mass as shown in Table VII. This is the reason why we have adopted

TABLE VI

Characteristic values of modes F_1, F_2, F_3

$10^{-5}\,\dfrac{M}{M_\odot}$	s (s^{-1}) F_1	F_2	F_3
1	9.776(−11)	±6.603(−6)i	−2.070(−10)
2	1.521(−10)	±3.832(−6)i	−3.167(−10)
3	3.635(−10)	±2.050(−6)i	−7.952(−10)
3.5	1.260(−9)	±1.025(−6)i	−4.497(−9)
3.68	1.196(−8)	±3.246(−7)i	−2.396(−8)
3.69	2.215(−8)	±2.374(−7)i	−4.441(−8)
3.70	6.185(−8)	±1.284(−7)i	−1.244(−7)
3.71	1.734(−7)	6.156(−8)	−2.350(−7)
3.72	2.921(−7)	2.687(−8)	−3.190(−7)
4	1.176(−6)	1.703(−9)	−1.178(−6)
5	2.221(−6)	3.918(−10)	−2.221(−6)
7	3.046(−6)	1.523(−10)	−3.046(−6)
10	3.591(−6)	7.837(−11)	−3.591(−6)

TABLE VII

Identification of modes F_1, F_2, F_3 with modes D_0^+, D_0^- and S_0

	$M < M_{cr}$	$M > M_{cr}$	
F_1	D_0^+	D_0^+	} unstable
F_2	D_0^-	S_0	} unstable
F_3	S_0	D_0^-	stable

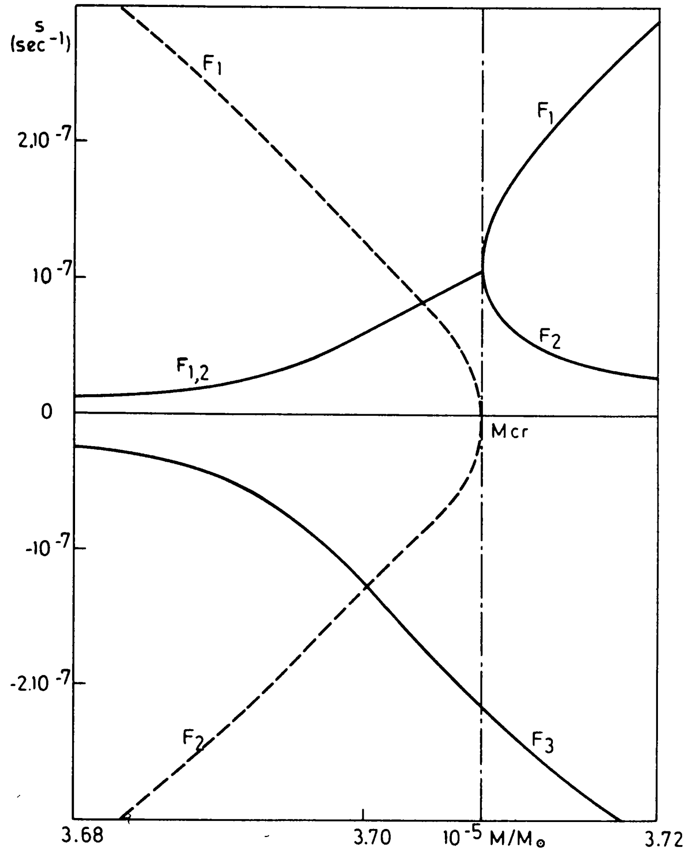

Fig. 8. Characteristic values s as a function of mass for modes F_1, F_2, F_3. Same convention as in Figure 2 for real and imaginary parts.

the new notations F_1, F_2, F_3. Figure 9 is a superposition of Figures 2 and 8. It shows clearly that a mode with a dynamical character for low values of the mass becomes progressively a secular mode as the mass increases (mode F_2) and reciprocally (mode F_3). Close to the critical mass it is no longer possible to distinguish between dynamical and secular fundamental modes. Both dynamical and secular terms play an important part in the determination of the three fundamental modes. Let us note that modes F_1 and F_2 are unstable on both sides of the critical mass and F_3 is stable in the whole range of mass. There does not appear any new unstable mode at the critical mass. Only the character of the instability changes through the critical mass. Below it modes F_1 and F_2 are vibrationally unstable and above F_1 is dynamically unstable and F_2 secularly unstable.

With regard to the harmonics of both types (dynamical and secular) the complete treatment does not bring about appreciable modifications. It is the damping coefficient of dynamical harmonics which is most affected and it differs only slightly from that obtained from Equation (23) as it is shown by the comparison of Tables IV and VIII. The discrepancy is only 1% for the first harmonic D_1 and increases with the order of the mode.

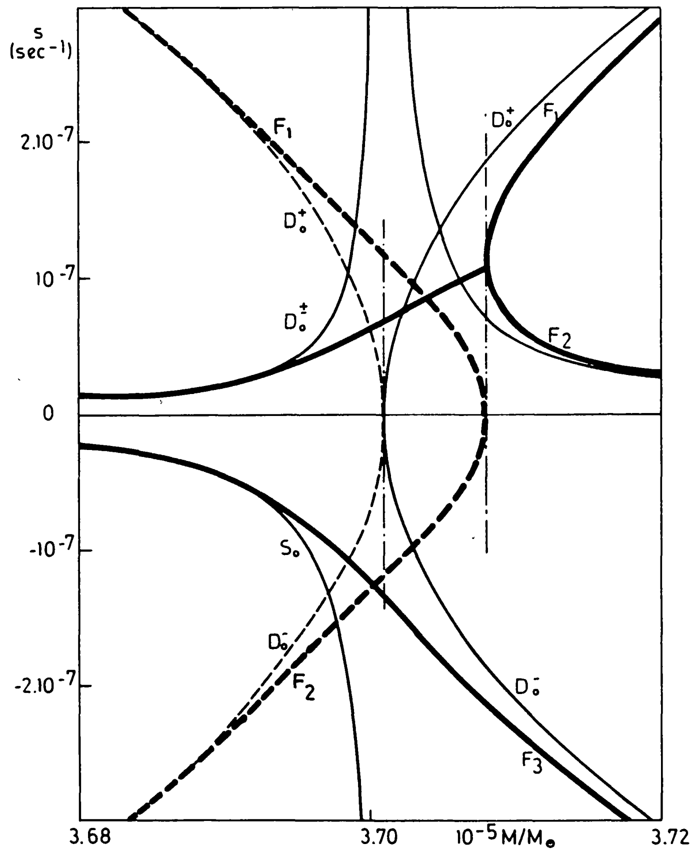

Fig. 9. Characteristic values s as a function of mass for fundamental modes computed in both approximations and in the complete treatment. The thicker lines refer to the complete treatment. Same convention as in Figure 2 for real and imaginary parts.

TABLE VIII

Characteristic values of the dynamical harmonics (complete treatment) for the model of 10^5 $M_\odot$

Mode	s (s^{-1})
D_1	$-3.494(-10) \pm 5.985(-5)i$
D_2	$-6.177(-9) \pm 8.552(-5)i$
D_3	$-4.462(-8) \pm 1.098(-4)i$
D_4	$-1.892(-7) \pm 1.336(-4)i$
D_5	$-5.472(-7) \pm 1.572(-4)i$
D_6	$-1.180(-6) \pm 1.808(-4)i$
D_7	$-2.043(-6) \pm 2.048(-4)i$
D_8	$-3.037(-6) \pm 2.291(-4)i$
D_9	$-4.081(-6) \pm 2.538(-4)i$
D_{10}	$-5.129(-6) \pm 2.788(-4)i$

9. Discussion

The hypothesis of absence of rotation and magnetic field limits the implications of our conclusions with respect to quasars. Our study confirms the results obtained previously about the stability of supermassive stars and clarifies their meaning. The part played by the instability in the evolution of the star is not clear. According to Osaki (1966) it involves the gravitational collapse of the star. However, according to other authors (Appenzeller and Fricke, 1972a, b; Fricke, 1973) there exists a range of mass in which the direction of the initial collapse is reversed by the nuclear reactions and the evolution ends with the disruption of the star.

With respect to the theory of radial perturbations of stars, we have brought out the existence of a continuous transition, in a sequence of varying mass, between modes of dynamical and secular types. This implies that the distinction in both types is not an absolute one. It rests only on the possibility to make the adiabatic approximation for dynamical modes and the quasi-static approximation for secular ones. When the secular characteristic time becomes of the same order of magnitude as the dynamical one, both approximations must be abandoned and the distinction is no longer significant. Particularly, when arguing about stability of a star, it is nonsense to consider cases where the secular time is shorter than the dynamical one. The same comment may be made about the different types of instabilities. Close to the critical mass the distinction between them becomes artificial. For instance a vibrational instability with an amplification time shorter than the period is not very different from a dynamical instability for the future evolution of the star. This last remark may explain a result obtained by Appenzeller and Fricke (1972a, b). Following the contraction of supermassive stars, they have found a sudden acceleration of the contraction before the star reaches the radius of dynamical instability. We suggest that this may be due to a vibrational instability with an amplification time shorter than the period. The inversion of the initial movement in one case supports this point of view.

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