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Properties of Non-radial Stellar Oscillations

by

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ABSTRACT

The properties of the second order linear eigenvalue problem describing the adiabatic non-radial oscillations of stars are discussed analytically. Cases with discontinuities in density are also considered. The distribution of amplitudes is studied using a simplified model which allows the interpretation of numerical results obtained for physical models.

1. Introduction

The mathematical properties of the eigenvalue problem describing the adiabatic non-radial oscillations of stars are still poorly known and our information rests only on numerical integrations.

Even when the problem is simplified neglecting the Eulerian perturbation of the potential a rigorous analysis of the eigenvalue problem is still lacking. As early as 1941 Cowling (1941) introduced the distinction between p and g spectra on the basis of an asymptotic discussion of the problem. Owen (1957) was unable to find the f-mode and the first p- and g-modes for polytropes of high central condensation. Robe (1968) showed that these modes still exist but that they acquire extra modes. Scuflaire (1974) and Osaki (1975) showed that a regularity can be found in all cases provided the nodes are counted in an appropriate way. We give here a rigorous discussion of the properties of the eigenvalue problem when the Eulerian perturbation of the potential is neglected. The discussion is also extended to cases where discontinuities in density are present in the star. For incompressible fluids, it is known that in such situations as many new modes, called discontinuity modes, as density discontinuities appear. We show that it is not always so in stars.

The results of this mathematical discussion are summarized in section 2.4.

When a density discontinuity is present in the star, there can nevertheless exist one or several modes having their largest amplitudes in the vicinity of one of the discontinuities. This problem is discussed using a simplified model whose predictions allow the interpretation of numerical results obtained from physical models (section 4).

2. Oscillatory Properties of Non-radial Oscillations

2.1. Equations and boundary conditions.

Neglecting the Eulerian perturbation of the gravitational potential (Cowling's approximation) the equation for non-radial oscillations are

$$\frac{dv}{dr} = aw, \tag{1}$$

$$\frac{dw}{dr} = bv, \tag{2}$$

with $v = f_1 r^2 \delta r$ and $w = f_2 p'/\varrho$,

$$f_1 = \exp\left(\int_0^r \frac{1}{\Gamma_1}\frac{d\ln p}{dr}\,dr\right), \tag{3}$$

$$f_2 = \exp\left(\int_0^r A\,dr\right), \quad A = \frac{d\ln\varrho}{dr} - \frac{1}{\Gamma_1}\frac{d\ln p}{dr}, \tag{4}$$

$$a = \left(\frac{\sigma_a^2}{\sigma^2} - 1\right)\frac{r^2}{c^2}\frac{f_1}{f_2}, \quad c^2 = \Gamma_1\frac{p}{\varrho}, \tag{5}$$

$$b = \frac{1}{r^2}(\sigma^2 - n^2)\frac{f_2}{f_1}, \tag{6}$$

where c is the velocity of sound, $n = \sqrt{-Ag}$ is the Brunt-Vaïsälä frequency, $\sigma_a = \sqrt{l(l+1)/r^2}\,c$ is the critical sound frequency, l is the degree of surface spherical harmonic.

Equations (1) and (2) are those given in Ledoux and Walraven (1958) modified to take the non-constancy of Γ_1 into account.

Equation (3) shows that f_1 is continuous throughout the star even when discontinuities in density are present. In such cases A must be considered as a distribution to maintain the validity of Eq. (4), f_2 is discontinuous at discontinuities of density and verifies the equation

$$\frac{f_{2+}}{\varrho_+} = \frac{f_{2-}}{\varrho_-}, \tag{7}$$

where the subscripts $-$ and $+$ refer to the lower and upper sides of the discontinuity.

For what follows it is useful to represent the solutions in the $(v(r), w(r))$ plane (Scuflaire 1974) and to introduce the polar coordinates (ψ, θ) defined by

$$v = \psi \cos\theta, \qquad w = \psi \sin\theta. \tag{8}$$

Then Eqs. (1) and (2) become

$$\frac{d\theta}{dr} = b\cos^2\theta - a\sin^2\theta, \tag{9}$$

$$\frac{d\psi}{dr} = (a+b)\psi \sin\theta\cos\theta. \tag{10}$$

The discussion of the properties of the eigenvalue problem is based on the behavior of the solutions of Eq. (9).

It is readily verified that the regularity condition at the center imposes that v and w go to zero respectively as r^{l+1} and r^l, and that

$$\lim_{r\to 0} rw/v = \sigma^2/l \quad \text{and} \quad \theta(0, \sigma^2) = \pi/2 + k\pi.$$

We may take $k = 0$ and we have

$$\theta(r, \sigma^2) = \frac{\pi}{2} - \frac{l}{\sigma^2} r, \tag{11}$$

for sufficiently small r.

The boundary condition to apply at the "surface" is less obvious especially for non-zero surface temperature models. In all cases we are led to a condition of the form $\theta(R) = \alpha + k\pi$, with $0 < \alpha < \pi/2$.

For zero surface temperature models the condition $\delta p(R) = 0$, which is equivalent to the condition of regularity of the solution, implies that

$$\theta = \mathrm{tg}^{-1}\left[\frac{GM}{R^4}\frac{f_2(R)}{f_1(R)}\right] + k\pi = \pi/2 + k\pi.$$

For these models it can be considered that in the outermost layers $(r > r_2)$, $m(r) = M$, $r \simeq R$ and $P = K\varrho^\gamma$ with γ constant. Then if $|n^2| \gg \sigma^2 \gg \sigma_a^2$ the regular solution is

$$\theta = k\pi + \mathrm{tg}^{-1}(\beta x^{-m}), \tag{12}$$

with $m = (2\gamma - \Gamma_1)/\Gamma_1(\gamma - 1)$, $x = R - r$, and

$$\beta = \frac{m + \sqrt{m^2 - 4a_1 b_1}}{2a_1}, \quad a_1 b_1 = \frac{\gamma(\gamma - \Gamma_1)}{\Gamma_1^2(\gamma - 1)^2}.$$

At a discontinuity in density δr and δp must be continuous. This implies the continuity of v and a discontinuity in w given by

$$w_+ - w_- = \frac{g}{r^2 f_1}\left(\frac{f_2}{\varrho}\right)(\varrho_+ - \varrho_-)v,$$

and for θ

$$\operatorname{tg}\theta_+ - \operatorname{tg}\theta_- = \frac{g}{r^2 f_1}\left(\frac{f_2}{\varrho}\right)(\varrho_+ - \varrho_-). \tag{13}$$

Obviously we may impose that $|\theta_+ - \theta_-| < \pi$ then θ_+ and θ_- belong to the same interval $I_k = [k\pi - \pi/2, k\pi + \pi/2]$.

2.2. Oscillatory Properties of $\theta(r)$.

We first discuss the behavior of $\theta(r)$ for a given value of σ^2. Let us consider for a given σ^2 a solution of Eq. (9), $\theta(r, \sigma^2)$, and an interval $\mathscr{R}_0 = [r_1, r_2]$ in which ϱ is continuous and a and b do not change sign. From Eqs. (1) and (2) one gets $\mathrm{d}(vw)/\mathrm{d}r = aw^2 + bv^2$.

If $ab > 0$ then vw varies monotonically and can have only one zero, i.e. only v or w can have a zero in $\mathscr{R}_0$. This means that if $ab > 0$ i.e. if $\max n^2(r) < \sigma^2 < \min \sigma_a^2(r)$, or if $\max \sigma_a^2(r) < \sigma^2 < \min n^2(r)$, or if $\sigma^2 < \min\{0, n^2(r)\}$, then $|\theta(r_2, \sigma^2) - \theta(r_1, \sigma^2)| < \pi$.

If $ab < 0$ then we can have $|\theta(r_2, \sigma^2) - \theta(r_1, \sigma^2)| > \pi$.

If $a < 0$ and $b > 0$, i.e. if $\sigma^2 > \max\{\sigma_a^2(r), n^2(r)\}$ or $\max n^2(r) < \sigma^2 < 0$, for $r \in \mathscr{R}_0$, then Eq. (9) shows that θ increases with r.

If $a > 0$ and $b < 0$, i.e. if $0 < \sigma^2 < \min\{\sigma_a^2(r), n^2(r)\}$ for $r \in \mathscr{R}_0$, then θ is decreasing with r.

If N discontinuities are present in $\mathscr{R}_0$ each discontinuity produces a discontinuity in θ smaller than π, keeping θ in the same interval I_k; therefore $|\theta(r_2, \sigma^2) - \theta(r_1, \sigma^2)| < (N+1)\pi$ when $ab > 0$.

Let us now consider θ as a function of σ^2.

Let $\theta_1(r) = \theta(r, \sigma_1^2)$ and $\theta_2(r) = \theta(r, \sigma_2^2)$, with $\sigma_2^2 > \sigma_1^2$, be the two solutions of Eq. (9) satisfying the central boundary condition (11). We first suppose ϱ continuous in $(0, r)$. We have:

$$\frac{\mathrm{d}}{\mathrm{d}r}(\theta_2 - \theta_1) = -(a_1 + b_1)(\sin\theta_2 + \sin\theta_1)\frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1} \times$$

$$\times(\theta_2 - \theta_1) + (b_2 - b_1)\cos^2\theta_2 + (a_1 - a_2)\sin^2\theta_2 = g(\theta_2 - \theta_1) + h.$$

Since a and b are decreasing and increasing functions of σ^2, respectively, $h(r) > 0$.

In the vicinity of $r = 0$, $\mathrm{d}(\theta_2 - \theta_1)/\mathrm{d}r > 0$ and $\theta_2 > \theta_1$. Therefore if $(\theta_2 - \theta_1)$ could have one zero for $r > 0$, we would have at that point $\mathrm{d}(\theta_2 - \theta_1)/\mathrm{d}r < 0$ which is impossible since $h > 0$. So $\theta_2 > \theta_1$ for $r > 0$, and θ is an increasing function of σ^2 for $\sigma^2 \in]-\infty, 0[$ and $\sigma^2 \in]0, \infty[$. This property remains true when crossing a discontinuity. $\theta_{2-} > \theta_{1-}$ and, if θ_{2-} and θ_{1-} belong to the same interval I_k, we have from Eq. (13): $\mathrm{tg}\theta_{2+} - \mathrm{tg}\theta_{2-} = \mathrm{tg}\,\theta_{1+} - \mathrm{tg}\theta_{1-}$ or $\mathrm{tg}\theta_{2+} - \mathrm{tg}\theta_{1+} = \mathrm{tg}\theta_{2-} - \mathrm{tg}\,\theta_{1-} > 0$. This means that $\theta_{2+} > \theta_{1+}$. If θ_{2-} and θ_{1-} do not belong to the same interval I_k the result is obvious.

Next we show that for $r > 0$, $\lim\limits_{\sigma^2 \to \infty} \theta(r, \sigma^2) = \infty$ (Proposition I). Let us consider an interval $\mathscr{R}_0 = [r_1, r_2]$ and $\sigma_1^2 = \max(\sigma_a^2(r), n^2(r))$ $(r \in \mathscr{R}_0)$. For any $\sigma^2 > \sigma_1^2$, $a < 0$ and $b > 0$ in $\mathscr{R}_0$ and θ increases with r. Let $m = \min(-a(r))$, $r \in \mathscr{R}_0$, $\sigma^2 > \sigma_1^2$ and $c = \min(b(a))$, $r \in \mathscr{R}_0$, $\sigma^2 > \sigma_1^2$. From Eqs. (1) and (2) we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{1}{a}\frac{\mathrm{d}V}{\mathrm{d}r}\right) + bV = 0. \tag{14}$$

Let us also introduce the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{m}\frac{\mathrm{d}V}{\mathrm{d}r}\right) + CV = 0. \tag{15}$$

The distance between consecutive zeros of $V(r)$ is $\pi/\sqrt{mC}$. From the comparison theorem (see *e.g.* Coddington and Levinson 1955) we know that $v(r)$ oscillates more rapidly than $V(r)$. Therefore if N is the number of zeros of $V(r)$ in $\mathscr{R}_0$ then $\theta(r_2, \sigma^2) > \theta(r_1, \sigma^2) + (N-1)\pi$. Since $\sqrt{mC}$ goes to infinity as σ,

$$\lim_{\sigma^2 \to \infty} N = \infty,$$

which proves the theorem.

We now show that for $r > 0$,

$$\lim_{\sigma^2 \to 0+} \theta(r, \sigma^2) = -\infty,$$

except if $n^2(r_0) < 0$ for all $r_0 < r$ (Proposition II). We first suppose $n^2(r) > 0$ for $r \in \mathscr{R}_0$ and let $0 < \sigma_1^2 = \min(\sigma_a^2(r), n^2(r))$, $(r \in \mathscr{R}_0)$.

For $0 < \sigma^2 < \sigma_1^2$, $a > 0$, and $b < 0$ in $\mathscr{R}_0$ hence θ is a decreasing function of r. Let

$$m = \min(-b(r)), \quad C = \min(a(r)), \quad r \in \mathscr{R}_0, \quad \sigma^2 < \sigma_1^2.$$

Following the same reasoning as above we obtain

$$\theta(r_2, \sigma^2) < \theta(r_1, \sigma^2) - (N-1)\pi,$$

where N is the number of zeros of the comparison equation in $\mathscr{R}_0$. Since $\sqrt{mC}$ goes to infinity as σ^2 goes to zero,

$$\lim_{\sigma^2 \to 0+} N = \infty.$$

If $n^2(r) < 0$ in $\mathscr{R}_0$, the result remains true as far as $n^2(r) > 0$ in some finite interval $\mathscr{R}_0' = [r_1', r_2']$ with $r_2' < r_2$. The result is valid for any point $r \in \mathscr{R}_0'$. In the region where $n^2(r) < 0$, $a > 0$ and $b > 0$ and θ can vary at most by π. We have

$$\theta(r_2, \sigma^2) < \theta(r_2', \sigma^2) + \pi,$$

and again

$$\lim_{\sigma^2 \to 0+} \theta(r_2, \sigma^2) = -\infty.$$

Let us now show that for $r > 0$,

$$\lim_{\sigma^2 \to 0_-} \theta(r, \sigma^2) = \infty,$$

except if $n^2(r_0) > 0$ for all $r_0 < r$ (Proposition III). Let us first suppose $n^2(r) < 0$, $r \in \mathscr{R}_0$, and let us define σ_1^2 such that

$$\max(n^2(r)) = \sigma_1^2 < 0, \quad r \in \mathscr{R}_0.$$

Then $a < 0$ for all $\sigma^2 < 0$ and $b > 0$ for $\sigma^2 > \sigma_1^2$; hence θ is an increasing function of r.
Let $m = \min(b(r))$, $r \in \mathscr{R}_0$, $\sigma^2 > \sigma_1^2$, $C = \min(-a(r))$, $r \in \mathscr{R}_0$, $\sigma^2 > \sigma_1^2$. Using again the comparison theorem we prove the theorem following the same reasoning as above.

In $n^2(r) > 0$ in $\mathscr{R}_0$ the theorem can be generalized as above and we have

$$\lim_{\sigma^2 \to 0-} \theta(r_2, \sigma^2) = +\infty,$$

if $n^2(r) < 0$ in some finite interval $\mathscr{R}_0' = [r_1', r_2']$ with $r_2' < r_2$.

Since θ varies through a discontinuity at most by π all these theorems remain valid when a finite number of discontinuities are present in the star.

2.3. Classification of the modes.

A. Positive σ^2.

If $\theta(r, \sigma^2)$ is a solution of Eq. (9) satisfying the central boundary condition (11), σ^2 is an eigenvalue if $\theta(R, \sigma^2)$ satisfies the surface boundary condition (12).

If $n^2(r)$ is not negative everywhere in the star, $\theta(R, \sigma^2)$ is a continuous function of σ^2 which varies from $-\infty$ to $+\infty$ when σ^2 increases from zero to infinity. As a result for each value of k there is σ_k^2 such that condition (12) is satisfied. The σ_k^2 obey the following properties:

(a) all σ_k^2 form an increasing series,

(b) $\lim\limits_{k \to -\infty} \sigma_k^2 = 0$,

(c) $\lim\limits_{k \to +\infty} x_k^2 = \infty$,

(d) the eigensolution $v_k(r)$ associated to σ_k^2 has exactly k zeros provided that the nodes are counted positively when $d\theta/dr > 0$ and negatively when $d\theta/dr < 0$.

It is easy to verify that p-modes correspond to $k > 0$ and g-modes to $k < 0$. The fundamental (f) mode is associated with $k = 0$.

Suppose we set the velocity of sound $c = \infty$, so $a > 0$ and proposition I cannot be proved. Moreover Eq. (11) shows that $\theta < \pi/2$ in the vicinity of the center and Eq. (9) shows that $\theta = \pi/2$ and $d\theta/dr > 0$ is impossible when $a > 0$. Nor can a discontinuity allow θ to jump through $\pi/2$ since as shown above θ_+ and θ_- are in the same interval I_k.

When $c = \infty$, modes with $k > 0$ do not exist and it is well known that this is also the case for acoustic modes (p-modes).

Suppose now that we consider a convectively unstable model so that $n^2(r) < 0$ for all $r > 0$, and $b > 0$ everywhere. Then proposition II cannot be proved and from Eq. (9) it is impossible to have both $\theta = k\pi$ and $d\theta/dr < 0$ when $b > 0$ and when ϱ is continuous. Therefore there exists no mode with $k < 0$ for fully convective models having no discontinuity in density. On the other hand, we know that such models have no g^+-modes.

Suppose now that there are N "stable" discontinuities, *i.e.* with $\varrho_+ < \varrho_-$ in the convectively unstable model. Each discontinuity can lead to a jump of θ through $k\pi$ ($k < 0$) since $\theta_+ < \theta_-$. We will now show that in such a situation there exist N g^+-modes. In this case b is positive everywhere and $a > 0$ in $\mathscr{R}_1 \equiv]\, 0, r_1 [$. For non zero surface temperature models $r_1 \equiv R$ for sufficiently small σ^2. For zero surface temperature models $r_1 < R$ since $\sigma_a(R) = 0$. But $(R - r_1)$ is proportional to σ^2 for σ^2 sufficiently small.

In $\mathscr{R}_1$, θ cannot go through $(k+1/2)\pi$ with $d\theta/dr > 0$ and through $k\pi$ with $d\theta/dr < 0$.

Writing Eq. (9) in the form

$$\frac{d\,\mathrm{tg}\,\theta}{dr} = a(\mathrm{tg}^2\,\theta_1 - \mathrm{tg}^2\theta),$$

with $\mathrm{tg}\,\theta_1 = \sqrt{b/a}$ and $0 < \theta_1 < \pi/2$, we see that $d\theta/dr < 0$ when $-(k+1)\pi + \theta_1 < \theta < -k\pi - \theta_1$, and is positive otherwise.

We also have:

$$\lim_{\sigma^2 \to 0+} a = \infty \quad \text{and} \quad \lim_{\sigma^2 \to 0+} \theta_1 = 0.$$

At the center $\theta = \pi/2$ and $d\theta/dr < 0$. Therefore, for sufficiently small σ^2, θ will rapidly decrease to values close to $+\theta_1$ but larger than 0. If σ^2 is small enough θ_1 is close to zero and θ jumps to values smaller than $-\theta_1$ when crossing the first discontinuity since $\theta_+ < \theta_-$ when $\varrho_+ < \varrho_-$. θ will then decrease to values close to $-\pi + \theta_1$, but larger than $-\pi$. The same behavior is reproduced for each discontinuity. After the last discontinuity θ converges towards values close to $(-N\pi + \theta_1)$.

At r_1, $\theta_1 = \pi/2$ but the interval over which θ_1 discards appreciably from zero is proportional to σ^{-2}. Since in $\mathscr{R}_1$ $d\theta/dr < b\cos^2\theta > 0$, then

$$\lim_{\sigma^2 \to 0+} \theta(r_1) = -N\pi.$$

In non zero surface temperature models $r_1 \equiv R$ for sufficiently small σ^2, then

$$\lim_{\sigma^2 \to 0+} \theta(R) = -N\pi.$$

Since θ is an increasing function of σ^2 the surface boundary condition will be fulfilled for σ^2 with $-1 \leqslant k \leqslant -N$ and we have N g^+-modes.

For zero surface temperature models

$$\lim_{\sigma^2 \to 0+} r_1 = R.$$

Moreover, n^2 and c^{-2} have singularities at the surface.

For $r > r_2$ we can assume that $m = M$, $r \simeq R$, and $p = K\varrho^\gamma$ with $\gamma =$ constant. Then:

$$\frac{f_1}{f_2} = K_1 x^m, \qquad m = \frac{2\gamma - \Gamma_1}{\Gamma_1(\gamma - 1)}, \qquad x = (R - r),$$

$$c^2 = \frac{GM}{R^2}\frac{\Gamma_1}{\gamma}(\gamma - 1)x, \qquad -n^2 = \frac{GM}{R^2}\frac{\gamma - \Gamma_1}{\Gamma_1(\gamma - 1)}\frac{1}{x}.$$

Let us introduce the comparison equation

$$\frac{d\theta_2}{dr} = N\cos^2\theta_2 + M\sin^2\theta_2. \tag{16}$$

with

$$M = (R^2/c^2)(f_1/f_3) = a_1 x^{m-1} > -a, \qquad N = a(-n^2/R^2)(f_2/f_1) = ab_1 x^{-m-1},$$

where a is taken large enough so that $N > b$. We have

$$\lim_{\sigma^2 \to 0} a = 1,$$

hence the solution of (16) which satisfies the surface boundary condition is

$$\theta_2 = k\pi + \operatorname{arctg} \beta_2 x^{-m},$$

with

$$\beta_2 = \frac{m + \sqrt{m^2 - 4aa_1b_1}}{2a}, \qquad a_1 b_1 = \frac{\gamma(\gamma - \Gamma_1)}{\Gamma_1^2(\gamma-1)^2}.$$

Let $\theta_s(r)$ be a solution of (9) satisfying the surface boundary condition. Near the surface $\mathrm{d}\theta_2/\mathrm{d}r > \mathrm{d}\theta/\mathrm{d}r$, as shown by Eq. (12). Therefore since $\theta_2(R) = \theta_s(R)$, we have $\theta_2(x) < \theta_s(x)$, and θ_s varies by less than $\pi/2$ in $[r_2, R[$. For σ^2 small enough, $r_2 < r_1$, and $\theta_1(r_2) = 0$ and $\theta(r_2) < \theta_s(r_2)$.

Since θ is an increasing function of σ^2, $\theta(r_2) = \theta_s(r_2)$ will be fulfilled with $-N \leqslant k \leqslant -1$ for N values of $\sigma^2 > 0$ which correspond to N g^+-modes.

The situation just considered may seem artificial but it is also of interest for homogeneous models with discontinuities.

It should be noticed that we find that $\sigma_{-1}^2 < \sigma_0^2 < \sigma_1^2$ for all l values. This result seems to be true in all models when the fourth order problem is considered, provided $l \geqslant 2$. For $l = 1$ the eigenvalue of the fundamental mode is then zero ($\sigma_0^2 = 0$) and the eigenfunction corresponds to a displacement of the whole star. This difference is due to our neglect of the perturbation of the potential which is a crucial approximation in that case.

B. Negative σ^2.

Let $\sigma_1^2 = \min(n^2(r))$, $r \in]0, R[$. Then $a < 0$ and $b < 0$, when $\sigma^2 < \sigma_1^2$. Hence for $\sigma^2 < \sigma_1^2$ and ϱ continuous, we have $\pi/2 < \theta(R, \sigma^2) < \pi$ because $\mathrm{d}\theta/\mathrm{d}r > 0$ at the center, and it is impossible to have $\theta = \pi$ and $\mathrm{d}\theta/\mathrm{d}r > 0$ or $\theta = \pi/2$ and $\mathrm{d}\theta/\mathrm{d}r < 0$ for $r > 0$. As a result it is impossible to fulfill the surface boundary condition (12) for any $\sigma^2 < \sigma_1^2$.

For models with at least one convectively unstable zone $(n^2(r) < 0)$ we have seen that

$$\lim_{\sigma^2 \to 0_-} \theta(R) = \infty.$$

Therefore in such models the surface boundary condition can be satisfied for an infinite set of eigenvalues σ_k^2, $1 < k < \infty$, and the corresponding eigenfunctions v_k have $(k-1)$ zeros outside the origin.

The σ_k^2 obey the following relation:

$$\sigma_k^2 > \sigma_1^2 \quad \text{and} \quad \lim_{k \to \infty} \sigma_k^2 = 0_- .$$

Clearly the spectrum is the g^--spectrum.

Let us now consider a fully radiative model ($n^2(r) > 0$ for all $r > 0$) which has N "unstable" discontinuities such that $\varrho_+ > \varrho_-$. We will show that there exist N g$^-$-modes. In such models for $\sigma^2 < 0$, a and b are negative, hence θ cannot go through $(k+1/2)\pi$ with $d\theta/dr < 0$ or through $k\pi$ with $d\theta/dr > 0$.

Writing Eq. (9) in the form $d\,tg\,\theta/dr = a(tg^2\theta_1 - tg^2\theta)$, with $tg\,\theta_1 = \sqrt{b/a}$, $0 < \theta_1 < \pi/2$, we see that $d\theta/dr > 0$ when $(k+1)\pi - \theta_1 > \theta > k\pi + \theta_1$, and is negative otherwise. We also have

$$\lim_{\sigma^2 \to 0_-} a = \infty \quad \text{and} \quad \lim_{\sigma^2 \to 0_-} \theta_1 = 0\,.$$

Hence since at the center $\theta = \pi/2$ and $d\theta/dr > 0$, for sufficiently small $|\sigma^2|$, θ will grow rapidly to values close to $(\pi - \theta_1)$ and then will remain close to that value.

For sufficiently small $|\sigma^2|$, θ_1 is small enough to allow θ to jump to values larger than $\pi + \theta_1$ when crossing the first discontinuity, since $\theta_+ > \theta_-$ when $\varrho_+ > \varrho_-$. θ will then increase to values close to $(2\pi - \theta_1)$ and the same scenario will be reproduced at each discontinuity. After the last discontinuity θ converges to values close to $[(N+1)\pi - \theta_1]$. θ is a continuous increasing function of σ^2 and

$$\lim_{r \to R} \theta_1 = \pi/2$$

under the same conditions as those required for $\alpha = \pi/2$ in Eq. (12), hence it is possible to find one σ^2 such that $|\sigma^2|$ is small but large enough to satisfy Eq. (12) with $k = N$. When $|\sigma^2|$ is increased Eq. (12) is satisfied for $1 \leqslant k \leqslant N$ and we have N g$^-$-modes.

2.4. Summary.

In the preceding pages we have proved in Cowling's approximation several properties of the eigenvalue spectrum of non-radial oscillations. They are based on an algebraic count of the nodes of δr. Nodes are assigned the sign of $d\theta/dr$ at that point (see Eq. (8) for the definition of θ).

These properties are:

1. All stars have a stable p spectrum of pressure modes, *i.e.* $\sigma_k^2 > 0$, $k = 1, 2, \ldots$, which has an accumulation point at infinity. The eigenfunction δr_k associated to σ_k^2 has k zeros.

2. There is a fundamental mode associated to $k = 0$, with $\sigma_0^2 > 0$. The algebraic sum of the nodes of δr_0 is equal to zero.

3. If the star has at least one radiative zone, there is a stable g$^+$ spectrum of gravity modes, i.e. $\sigma_k^2 > 0$, $k = -1, -2, \ldots$, which has an accumulation point at zero. The eigenfunction δr_k associated to σ_k^2 has k zeros ($k < 0$).

4. If the star has at least one convectively unstable zone there is an unstable g^- spectrum of gravity modes, sometimes called convective modes, *i.e.* $\sigma_k^2 < 0, k = +1, +2, \ldots,$ which has an accumulation point at zero. The δr_k associated to σ_k^2 has k zeros. The smaller eigenvalue σ_1^2 is larger than the minimum of the square of Brunt-Väisälä frequency.

5. If there are N "unstable" discontinuities in the star such that the density below the discontinuity is smaller than above, then N new modes appear in the eigenvalue spectrum provided there is no convectively unstable zone, *i.e.* if there is no g^- spectrum. We call these modes unstable discontinuity modes. They are all unstable *i.e.*; $\sigma_j^2 < 0, j = 1, \ldots, N$.

6. If there are N "stable" discontinuities in the star such that the density below the discontinuity is larger than above, then N new modes appear in the eigenvalue spectrum provided there is no g^+ spectrum. (The homogeneous model is an example of such a situation.) We call these modes stable discontinuity modes. They are all stable *i.e.* $\sigma_j^2 > 0, j = 1, \ldots$ $\ldots, N$.

3. Behaviour of Eigenfunctions at a Discontinuity

3.1. Analytical Discussion.

From the results of the preceding discussion we see that under some circumstances the presence of discontinuities in density introduce new modes which may be named discontinuity modes while under other (and most common) circumstances no new mode may be associated with a discontinuity. This of course leaves the problem of the behaviour of the eigenfunctions unsolved and it may well be that some modes show large amplitude near the discontinuity. Numerical computations show that it is so.

Some properties of these modes can be understood with the help of a simplified model which can be studied analytically.

Let us consider a system formed of 4 zones labeled 1 to 4 from bottom to top. We will suppose a plane geometry and that the gravity g, the sound velocity c and the Brunt-Väisälä frequency n are constant. In zones 1 and 4 the perturbations are supposed to be sinusoidal; vertical wave number is γ_i, $i = 1$ or 4. Zones 2 and 3 are separated by a discontinuity in density. In order for the amplitude δz at the discontinuity to be much larger than in regions 1 or 4 the perturbations must have an exponential behaviour in zones 2 and 3 of the type $\exp(\pm\lambda_i z)$, $i = 2$ or 3. This implies (Tolstoy 1963) that

$$\lambda_i^2 = k_H^2 + \nu^2 - \frac{\sigma^2}{c^2} - \frac{kH^2 n^2}{\sigma^2} > 0,$$

where k_H is the horizontal wave number and

$$\nu = -\frac{1}{2}\frac{\mathrm{d}\ln\varrho}{\mathrm{d}z}.$$

After some calculations it is possible to get the ratio R_i, $i = 1$ or 4 of the amplitude at the discontinuity to the amplitude in zone 1 or 4.

It is found that in order to have $R_1 \gg 1$ and $R_4 \gg 1$ σ^2 must be close to

$$\sigma_d^2 = g k_H^2 \frac{\varrho_2 - \varrho_3}{\varrho_2 \lambda_2 + \varrho_3 \lambda_3 - \nu(\varrho_2 - \varrho_3)}. \tag{17}$$

This expression is very similar to that obtained for incompressible fluids (see for instance Landau and Lifchitz 1959). The width of the domain of σ^2 around σ_d^2 where $R_1 > 1$ decreases when $h_2\lambda_2$ increases (h_2 is the thickness of the second zone). Very often $h_2\lambda_2 \gg 1$ and then the width of that domain is given by Eq. (18).

$$\left|\frac{\sigma^2 - \sigma_d^2}{\sigma_d^2}\right| < \frac{\lambda_1}{\sqrt{\gamma_1^2 + \lambda_2^2}} \frac{\lambda_2 \varrho_2 + \lambda_3 \varrho_3}{\lambda_2 \varrho_2 + \lambda_3 \varrho_3 - \nu(\varrho_2 - \varrho_3)} \exp(-h_2\lambda_2). \tag{18}$$

A similar condition can be obtained to ensure $R_4 > 1$.

In this local analysis we have not taken into account the conditions which the modes must satisfy at the boundaries of the star (at the centre and at the surface). When these are taken into account, a set of discrete frequencies for the modes are obtained. Depending upon the model, the spectrum of frequencies will be more or less dense in the vicinity of σ_d and several or only one mode will exhibit large amplitude at the discontinuity. The probability to have several modes with large amplitudes at the discontinuity increases when σ_d goes in the range of g or p modes of larger and larger order.

When we consider the case of two homogeneous incompressible layers, there is a unique mode owing its existence to the presence of this discontinuity. It is important to emphasize that in more general situations, there is no longer mode owing its existence to the presence of the discontinuity. The effect of the discontinuity is to change the behaviour of modes whose frequencies lie in an interval determined by the discontinuity.

3.2. Numerical Calculations.

The predictions of this simplified theory have been checked on stellar models of a 1.1 $M_\odot$ star with a chemical composition given by $X = 0.6$, $Z = 0.044$. At the beginning of the main sequence phase for $X_c > 0.25$

the models have a small growing convective core with a "stable" discontinuity developing at the top of the core. Numerical calculations have been performed for spherical harmonic degree l equal to 10, 25, 50 and 100. In this situation there is no new mode associated with the discontinuity. Nevertheless some modes, sometimes one, sometimes several, depending upon the importance of the discontinuity and the chosen l value, have their largest amplitude on the discontinuity and we will also call them discontinuity modes.

Table 1

A few properties of the models. q is the mass fraction and x the fractional radius at the discontinuity. X_c is the central hydrogen abundance

n°	$\frac{\varrho_1 - \varrho_2}{\varrho_1 + \varrho_2}$	q	x	X_c
1	3.8×10^{-3}	3.41×10^{-2}	7.13×10^{-2}	0.568
2	2.3×10^{-2}	4.20×10^{-2}	7.15×10^{-2}	0.466
3	6.2×10^{-2}	4.83×10^{-2}	6.9×10^{-2}	0.332

Table 2a

Dimensionless eigenvalues ω^2 of discontinuity modes, ratio R_a of amplitudes on the discontinuity to the maximum value in the rest of the star and identification of the modes for model 1

l	10	25	50	100
ω^2	8.747	19.682	38.256	75.42
R_a	11.31	10^7	8×10^6	10^5
	g_6	g_4	f	f
ω^2	8.626			
R_a	78			
	g_7			
ω^2	7.478			
R_a	2.10			
	g_8			

A few properties of the models studied are given in Table 1. Table 2 gives the eigenvalues ω^2 of the discontinuity modes in unit GM/R^3, the ratio R_a of the amplitude of $\delta r/r$ on the discontinuity of its largest extremum value elsewhere in the star and the identification of the modes. In all cases except one, only one discontinuity mode was found. For model 1 and $l = 10$ three discontinuity modes were found but their amplitude on the discontinuity is much smaller than in the other cases. The other eigenvalues show R_a values much smaller than one.

It must be noticed that for all eigenvalues

$$n_d^2 < \sigma^2 < \sigma_{a,d}^2,$$

where the subscript d indicates that n^2 and σ_a^2 are computed at the upper side of the discontinuity. Therefore these modes are by no way comparable to the modes sometimes trapped in the interior for which $\sigma^2 < n^2$. In order to have similar trapped modes it would be necessary to replace the discontinuity by a sharp ϱ gradient. But then a complete spectrum may appear in appropriate circumstances, for instance if, in a fully radiative model, an unstable discontinuity is replaced by a ϱ gradient.

Table 2b

Same as table 2a but for model 2

l	10	25	50	1000
ω^2	51.38	120.1	233.8	454.2
R_a	4×10^6	2×10^{23}	4×10^{50}	4×10^{47}
	p_2	p_3	p_2	p_2

Table 2c

Same as table 2a but for model 3

l	10	25	50	100
ω^2	145.0	340.5	666.2	1317
R_a	3×10^7	5×10^{22}	3×10^{49}	6×10^{44}
	p_5	p_6	p_6	p_6

The value predicted by Eq. (17) lies always within a factor 2 of the figures in Tables 2. It is systematically too small but comes closer to the numerical values as l or $(\varrho_1-\varrho_2)/(\varrho_1+\varrho_2)$ increases. Eq. (17) gives therefore a useful order of magnitude for the search of the eingenvalues of the discontinuity modes.

All the modes have an exponential behaviour in the vicinity of the discontinuity with $\lambda r = l$. Therefore the simplified model predicts that the maximum possible amplitude on the discontinuity increases with l while the peak sharpens. This behaviour is found in the numerical results. Firstly, R_a tends to increase with l. Secondly, 3 discontinuity modes are found for the lowest l in model 1 and these modes fall among high order g-modes.

For a given l value, when the discontinuity mode is a g-mode, the eigenfunction will cease to have an exponential type behaviour for smaller r/R than when it is a p-mode. To interpret our result with the aid of

Eq. (18) this means that $h_2 \lambda_2$ is smaller in the first case than in the second. This means that it is more likely to have only one discontinuity mode in the second situation than in the first one.

The very peculiar amplitude distribution of gravity modes should be kept in mind when discussing the stability of models with discontinuities in density.

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