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DISTRIBUTION OF ENERGY IN STELLAR NON-RADIAL OSCILLATIONS

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SUMMARY

We consider the kinetic energy and the potential energy of a non-radial adiabatic oscillation of a star. The kinetic energy can be divided into a fraction associated to the radial motion and a fraction associated to the horizontal motion and the potential energy can be decomposed into several terms, each one being related to a force involved in the problem (restoring force due to the compressibility of the gas, buoyancy, gravity). We use these different terms to describe the non-radial modes of a star.

1. INTRODUCTION

The nature of the spectrum of the non-radial adiabatic oscillations of a star is presently rather well understood. In the Cowling approximation (i.e. when the perturbation of the gravitational potential is neglected), the classification proposed by Scuflaire (1974) and Osaki (1975) has recently received a mathematical basis (Gabriel, Scuflaire, 1979). But, except for simple models such as polytropic models with low central condensation, this classification tells nothing about the physical properties of the modes. In the present paper we use the different terms of a particular decomposition of the energy of the modes to describe their physical properties. In the following only adiabatic perturbations are considered.

2. BASIC EQUATIONS

The basic differential equations governing the first order perturbations of an equilibrium model of star are well-known (see for instance Ledoux, Walraven, 1958). The following form of the energy integral may be derived from them through standard algebraic manipulations. We have written X′ and δX respectively for the eulerian (local) perturbation and the lagrangian perturbation of any variable X.

$$\int \left(\frac{1}{2}\rho v'^2 + \frac{P'^2}{2\rho c^2} + \frac{1}{2}\rho n^2 \delta r^2 + \frac{1}{2}\rho'\Phi'\right) dV = E = \text{Const.} \qquad (1)$$

The domain of integration may be limited to the volume of the star as the integrand vanishes outside the star. The symbols have their usual meanings :

v is the velocity

ρ the density (i.e. the mass per unit volume)

P the pressure

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c the speed of sound

δr the radial component of the displacement

n the Brunt-Väisälä frequency, given by

$$n^2 = -Ag \text{ with } A = \partial_r \ln \rho - \partial_r \ln P/\Gamma_1$$
$$\Gamma_1 = (\partial \ln P/\partial \ln \rho)_S$$

Φ the gravitational potential defined in such a way that the gravitational force $\mathbf{g}$ exerted on the unit mass is given by $g_i = - \partial_i \Phi$.

The different terms in equation (1) are respectively the kinetic energy E_K and potential energy terms related to the restoring force due to the compressibility of the gas (as in acoustic waves) E_A, to the buoyancy E_B and to the gravitational field E_G

$$E_K = \int \frac{1}{2} \rho v'^2 \, dV$$

For our purpose it is useful to decompose further this term in a part related to the radial component of the velocity v'_r and a part related to the horizontal component of the velocity v'_h

$$E_K = E_{Kr} + E_{Kh}, \text{ with}$$
$$E_{Kr} = \int \frac{1}{2} \rho v_r'^2 \, dV$$
$$E_{Kh} = \int \frac{1}{2} \rho v_h'^2 \, dV$$

The three terms of potential energy write

$$E_A = \int \frac{P'^2}{2\rho c^2} dV$$
$$E_B = \int \frac{1}{2} \rho n^2 \delta r^2 dV$$
$$E_G = \int \frac{1}{2} \rho' \Phi' \, dV$$

Using Poisson's equation, it is easy to see that the last term can also be written as

$$E_G = - \int \frac{g'^2}{8\pi G} dV$$

the integration now being performed through the whole space (for this reason we prefer the first form for numerical computation).

With these notations Eq. (1) writes

$$E_K + E_A + E_B + E_G = E = \text{Const.} \tag{2}$$

The first two terms are positive.

$$E_K > 0, \quad E_A > 0.$$

The term E_B can assume both signs. For instance, in a wholly radiative star $E_B > 0$

($n^2 > 0$ everywhere), whereas in a wholly convective star $E_B < 0$ ($n^2 < 0$ everywhere). E_G is always negative

$$E_G < 0.$$

3. APPLICATION TO NON-RADIAL MODES OF OSCILLATION

Let us rapidly redefine the notations used to describe a non-radial mode of oscillation of a star. If r, θ, φ are the spherical coordinates of a point of the star and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ unit vectors in the directions of increasing r, θ, φ respectively, we define

$$\mathbf{a}_r(\theta, \varphi) = \sqrt{4\pi}\, Y_{lm}(\theta, \varphi)\mathbf{e}_r$$

$$\mathbf{a}_h(\theta, \varphi) = \sqrt{\frac{4\pi}{l(l+1)}} \left[\partial_\theta\, Y_{lm}\mathbf{e}_\theta + \frac{1}{\sin\theta}\, \partial_\varphi\, Y_{lm}\mathbf{e}\right]$$

where the $Y_{lm}(\theta, \varphi)$'s are the spherical harmonics in their real form (i.e. with factors $\cos m\varphi$ and $\sin m\varphi$ rather than $\exp(im\varphi)$). The factor $\sqrt{4\pi}$ has been introduced so that we have

$$\iint |\mathbf{a}_r|^2 \sin\theta\, d\theta d\varphi = \iint |\mathbf{a}_h|^2 \sin\theta\, d\theta d\varphi = 4\pi.$$

The displacement $\delta\mathbf{r}$ in a non-radial mode of angular frequency σ can be written

$$\delta\mathbf{r}(r, \theta, \varphi, t) = [\xi_r(r)\mathbf{a}_r(\theta, \varphi) + \xi_h(r)\mathbf{a}_h(\theta, \varphi)]\cos\sigma t$$

where $\xi_r(r)$ and $\xi_h(r)$ are functions of r only, describing respectively the radial part and the horizontal part of the displacement.

From the above definitions we have

$$\int \left|\frac{\delta\mathbf{r}}{R}\right|^2 \rho dV = \int 4\pi r^2\rho\, \xi^2 dr$$

with $\xi^2 = \xi_r^2 + \xi_h^2$.

Using the dimensionless variables y, z, u, v as in Boury et al. (1975) (the factor $\sqrt{4\pi}$ was however omitted) we have

$$\xi_r = x^{l-1}y$$

$$\xi_h = x^{l-1}\frac{\sqrt{l(l+1)}}{\omega^2}\left(u + \frac{RP}{GM\rho}\, z + \frac{q}{x^3}\, y\right)$$

with

$$\omega^2 = \frac{R^3\sigma^2}{GM}$$

$$x = r/R$$

$$q = m/M.$$

We have computed adiabatic non-radial modes of oscillation with the same set of differential equations as Boury et al. (1975). The expressions of some other variables may be useful :

$$\Phi' = \frac{GM}{R}\, x^l u \sqrt{4\pi}\, Y_{lm}(\theta, \varphi)\cos\sigma t$$

$$\frac{P'}{\rho} = \frac{GM}{R} x^l \left(\frac{q}{x^3} y + \frac{RP}{GM\rho} z \right) \sqrt{4\pi}\, Y_{lm}(\theta, \varphi) \cos \sigma t$$

$$\frac{\delta P}{P} = x^l z \sqrt{4\pi}\, Y_{lm}(\theta, \varphi) \cos \sigma t.$$

The terms of Eq. (2) take the forms

$$E_K = E_K^0 \sin^2 \sigma t$$

$$E_A = E_A^0 \cos^2 \sigma t,\ E_B = E_B^0 \cos^2 \sigma t,\ E_G = E_G^0 \cos^2 \sigma t$$

$$E_K^0 \sin^2 \sigma t + (E_A^0 + E_B^0 + E_G^0) \cos^2 \sigma t = E.$$

So we have

$$E_K^0 = E_A^0 + E_B^0 + E_G^0 = E. \tag{3}$$

The same relation holds for the mean values

$$\bar{E}_K = \bar{E}_A + \bar{E}_B + \bar{E}_G$$

These relations merely express the equipartition of energy between kinetic and potential forms. It may be useful to express E_K^0, E_A^0, E_B^0 and E_G^0 in terms of the variables y, z, u :

$$E_K^0 = \frac{1}{2} \frac{GM^2}{R} \omega^2 \int \xi^2 \, dq$$

$$E_{Kr,h}^0 = \frac{1}{2} \frac{GM^2}{R} \omega^2 \int \xi_{r,h}^2 \, dq$$

$$E_A^0 = \frac{1}{2} \frac{GM^2}{R} \int \frac{GM\rho}{\Gamma_1 RP} x^{2l} \left(\frac{q}{x^3} y + \frac{RP}{GM\rho} z \right)^2 dq$$

$$E_B^0 = -\frac{1}{2} \frac{GM^2}{R} \int \frac{q}{x^3} \frac{RA}{x} x^{2l} y^2 \, dq$$

$$E_G^0 = \frac{1}{2} \frac{GM^2}{R} \int \left[\frac{z}{\Gamma_1} - y \left(\frac{RA}{x} - \frac{GM\rho}{\Gamma_1 RP} \frac{q}{x^3} \right) \right] u x^{2l} \, dq.$$

Our Eq. (3) may be derived from Chadrasekhar's Eq. (14) (Chandrasekhar, 1964). With our notations, Chandrasekhar's equation is written

$$\sigma^2 \int \rho \, |\, \boldsymbol{\delta r} \,|^2 dV = \int \left(\partial_i P' - \frac{\rho'}{\rho} \partial_i P + \rho \partial_i \Phi' \right) \delta x_i dV. \tag{4}$$

The right-hand side is transformed using the continuity and the adiabatic equations :

$$\left(\partial_i P' - \frac{\rho'}{\rho} \partial_i P + \rho \partial_i \Phi' \right) \delta x_i = \partial_i [(P' + \rho \Phi') \delta x_i] + \frac{P'}{\rho c^2} + \rho n^2 \delta r^2 + \rho' \Phi'.$$

We integrate this equality over a sphere of radius r greater than the radius of the star and the right-hand side of Eq. (4) takes the form

$$\int_S (P' + \rho \Phi') \delta x_i dS_i + \int_S \left(\frac{P'^2}{\rho c^2} + \rho n^2 \delta r^2 + \rho' \Phi' \right) dV.$$

The surface integral vanishes and we obtain

$$\sigma^2 \int \rho \mid \delta \mathbf{r} \mid^2 dV = \int \left(\frac{P'^2}{\rho c^2} + \rho n^2 \delta r^2 + \rho' \Phi'\right) dV$$

equivalent to our Eq. (3).

4. DESCRIPTION OF NON-RADIAL MODES

The different terms of energy we have written may be used to provide a physically meaningful description of a mode. In this section we suggest some ways of describing a mode. Three star models of increasing complexity are used to illustrate our point of view. Model 1 is the polytrope of index $n = 3$ and $\Gamma_1 = 5/3$. Its central condensation is low ($\rho_c/\bar{\rho} = 54.18$). Model 2 is the polytrope of index $n = 4$ and $\Gamma_1 = 5/3$. Due to the higher central condensation ($\rho_c/\bar{\rho} = 622.4$), extra nodes appear in the lower order modes. For our purpose, the structure of the models are best described by the cut-off frequencies for the radial propagation of acoustic waves (σ_a) and gravity waves (σ_g) as functions of the radius :

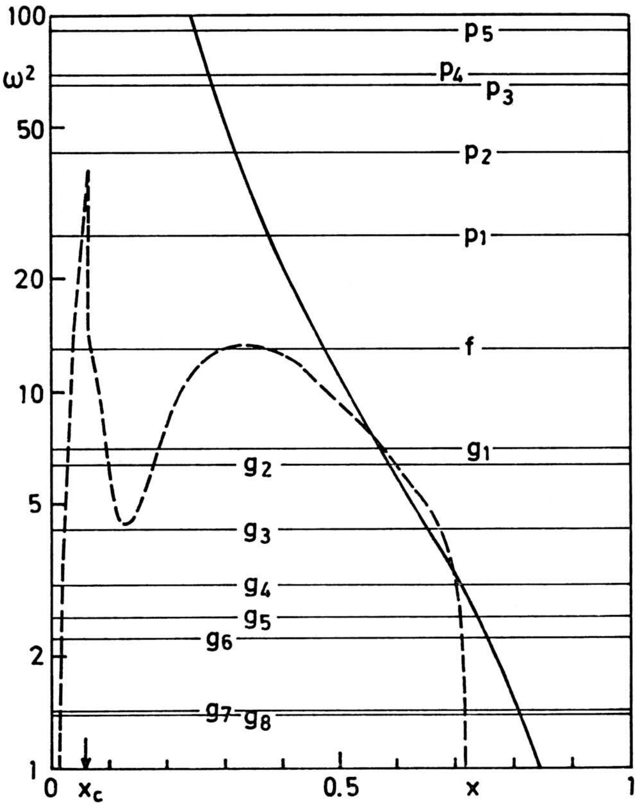

Fig. 1. — The solid curve and the dashed curve give respectively ω_a^2 ($l = 2$) and ω_g^2 as functions of the fraction radius x. The horizontal straight lines give ω^2 for some $l = 2$ non-radial modes. The vertical arrow shows the position of the discontinuity

$$\sigma_a = \sqrt{l(l+1)}\,\frac{c}{r}$$

$$\sigma_g = n.$$

The graphs of these functions are given for these models (and for $l = 2$) in Figs. 4 and 5 of Scuflaire (1974).

Model 3 is a physical model of 1 $M_\odot$ with a density discontinuity (due to a change in composition) located at $x_c = 0.0615$ ($q_c = 0.03$) such that $\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = 0.32$ where ρ_1 and ρ_2 are the densities respectively just below and just above the discontinuity. The ratio of central density to mean density is $\rho_c/\bar{\rho} = 168.3$. Further details on this model can be found in Boury et al. (1980) where it is called model 1. σ_a and σ_g as functions of the radius are given in dimensionless form in Fig. 1.

$$\omega^2_{a,g} = \frac{R^3\sigma^2_{a,g}.}{GM}$$

The simplest but rather crude way to describe a non-radial mode is to give the fractions of its kinetic energy in the radial and in the horizontal components of the motion. We define the fractions

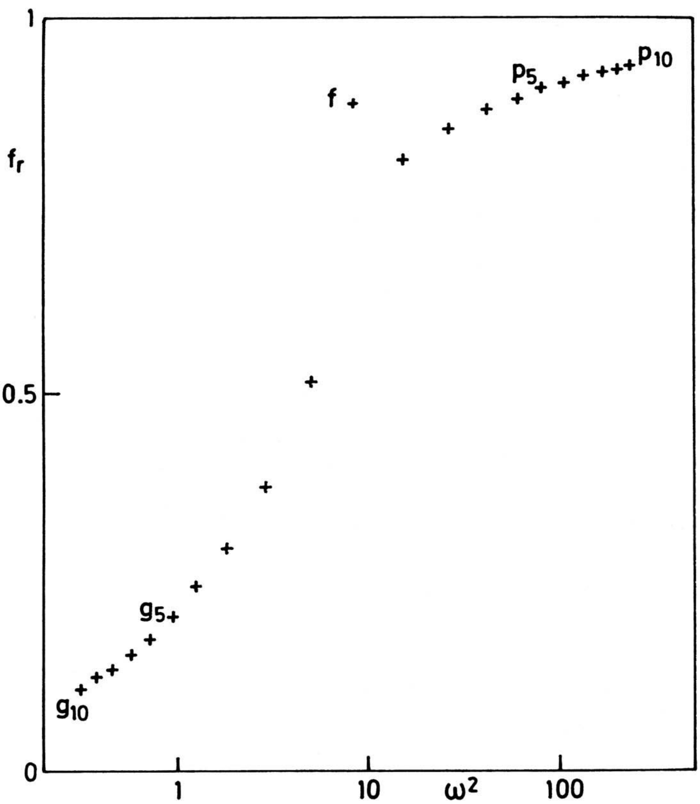

Fig 2. — The fraction f_r of the kinetic energy in the radial component of the movement is given versus ω^2 for some $l = 2$ modes of model 1 (polytrope $n = 3$, $\Gamma_1 = 5/3$).

$$f_r = \frac{E^0_{Kr}}{E}, \quad f_h = \frac{E^0_{Kh}}{E}$$

with $f_r + f_h = 1$

$$0 \leqslant f_r, \quad f_h \leqslant 1.$$

Fig. 2 gives f_r for some non-radial modes of degree $l = 2$ for model 1. This figure expresses quantitatively the observation made by Cowling (1941) that in g modes the motion is essentially horizontal whereas in p modes it is essentially radial. The same figure for model 2 (Fig. 3) shows essentially the same increase

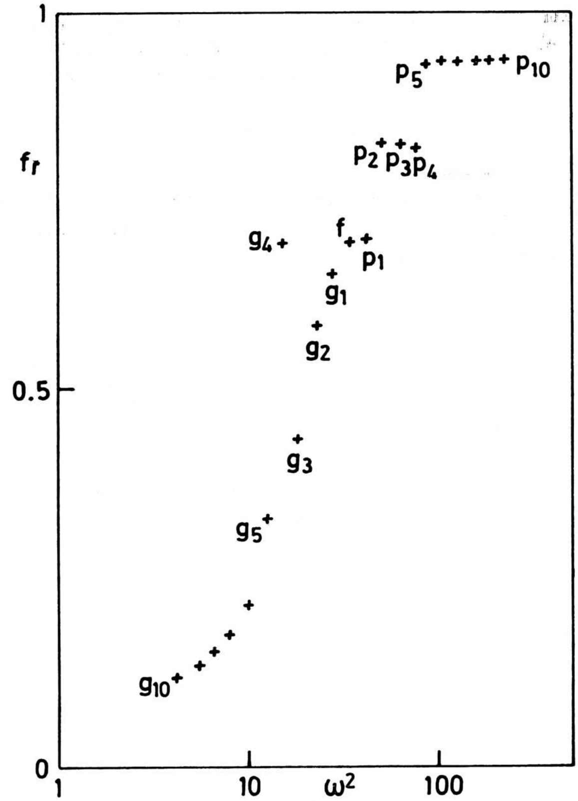

Fig. 3. — Same as Fig. 2, for model 2 (polytrope $n = 4$, $\Gamma_1 = 5/3$).

of f_r when going from g modes to p modes with irregularities corresponding to the apparition of extra-nodes in the eigenfunctions (see for instance Scuflaire, 1974). Fig. 4 gives some $l = 2$ modes of model 3. The irregularities suggest that this way of describing a non-radial mode is too crude for a rather complicated physical model. We shall see later that the non-radial modes of this model can be separated into different families on the basis of another criterion.

Instead of using f_r we can use f_A, f_B, f_G defined as

$$f_A = \frac{E^0_A}{E}, \quad f_B = \frac{E^0_B}{E}, \quad f_G = \frac{E^0_G}{E}.$$

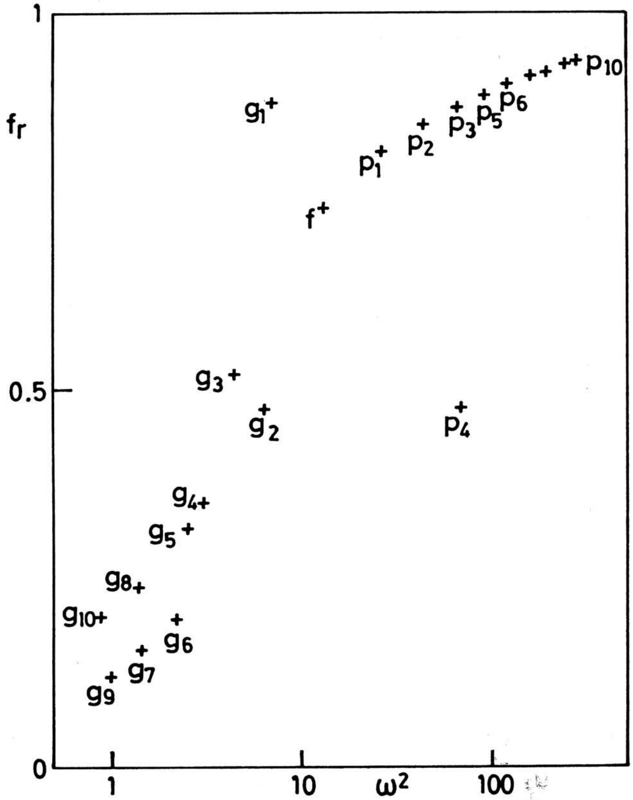

Fig. 4. — Same as Fig. 2, for model 3 (physical model).

For simple models f_A is large for p modes. This corresponds to the well-known fact that the pressure force plays a dominant role for these modes. On the reverse for g modes the main restoring force is buoyancy and f_B is large. For a more complex model the values of f_A and f_B vary quite irregularly with the sequence number of the mode, as f_r does.

The type of description already used by Goossens and Smeyers (1974) will give us a deeper insight into the physical characteristics of a non-radial mode. These authors considered star models composed of three distinct zones. For a given non-radial mode they specified the fraction of σ^2 (written as an integral expression of the eigenfunction from Eq. 4) contributed by each zone. We have applied here the same idea, using the kinetic energy because of its direct physical meaning. We define f_C as the fraction of the kinetic energy in the core

$$f_C = \int_0^{q_c} \xi^2 dq \Big/ \int_0^1 \xi^2 dq.$$

In fact we can omit the denominator, as our eigenfunctions are normalized in such a way that

$$\int \xi^2 dq = 1.$$

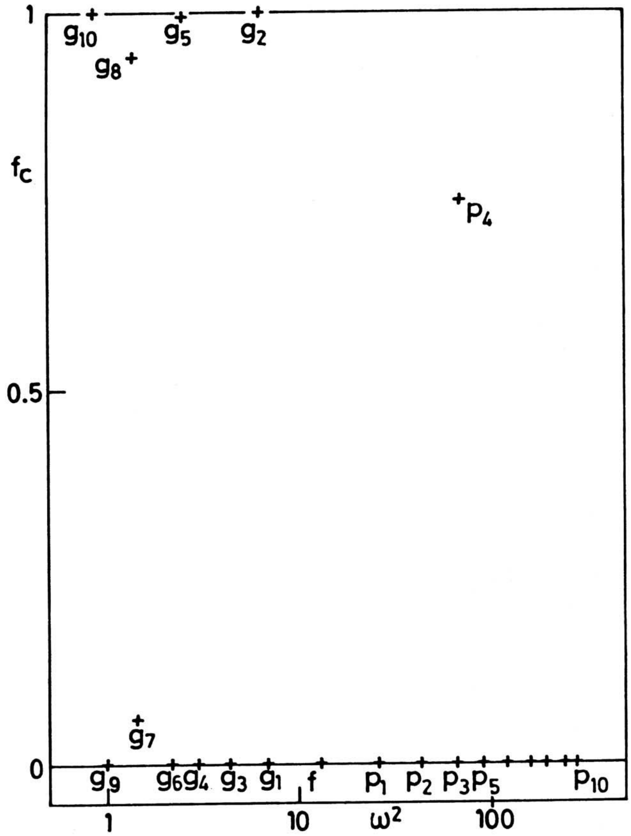

Fig. 5. — Fraction of the kinetic energy in the core f_C versus ω^2 for some $l = 2$ modes of model 3.

Fig. 5 shows that the non-radial $l = 2$ modes of model 3 can be separated into two families : modes having most of their energy in the core (g_{10}, g_8, g_5 and g_2) and modes having most of their energy in the envelope (the other ones, except p_4 which will be discussed below). Figs. 6 and 7 show both components of the displacement for one mode of both families. The $l = 2$ p_4 mode has an important fraction of its kinetic energy in the core. Figs. 8 and 9 show the displacement for this mode and a neighbouring one (p_5). The mode p_4 ($l = 2$) is a discontinuity mode associated to the discontinuity of density (Gabriel, Scuflaire, 1979). Its energy is confined in regions close to the discontinuity, i.e. the boundary of the core.

The distribution of energy of this mode has the following particularity. For a given mode let us distinguish three zones in a star : an A zone defined by $\sigma > \sigma_a, \sigma_g$ (zone of acoustic wave propagation), a G zone defined by $\sigma < \sigma_a, \sigma_g$ (zone of gravity wave propagation) and an E zone where none of these conditions is satisfied (evanescent zone as called by Osaki, 1975). Let f_E be the fraction of kinetic energy in the E zone. Fig. 10 shows that a distinctive character of the discontinuity mode is the localization of its energy in the evanescent zone (f_E equals 99.9 % for this mode). The study of the fractions of energy in the A and G zones leads to a distinction between modes of acoustic or gravity types.

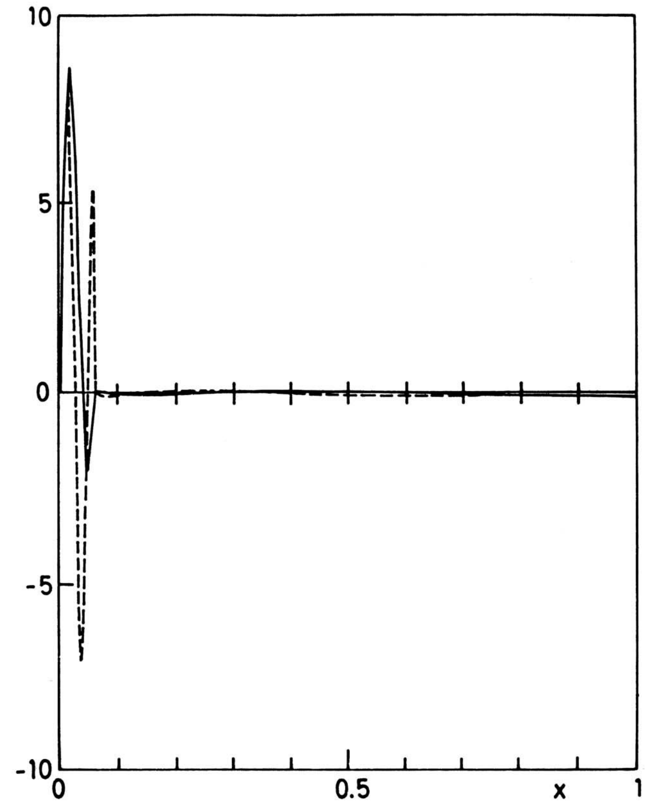

Fig. 6. — Radial component ξ_r (solid curve) and horizontal component ξ_h (dashed curve) of the displacement for $l = 2$ mode g_5 of model 3 (a mode of oscillation of the core).

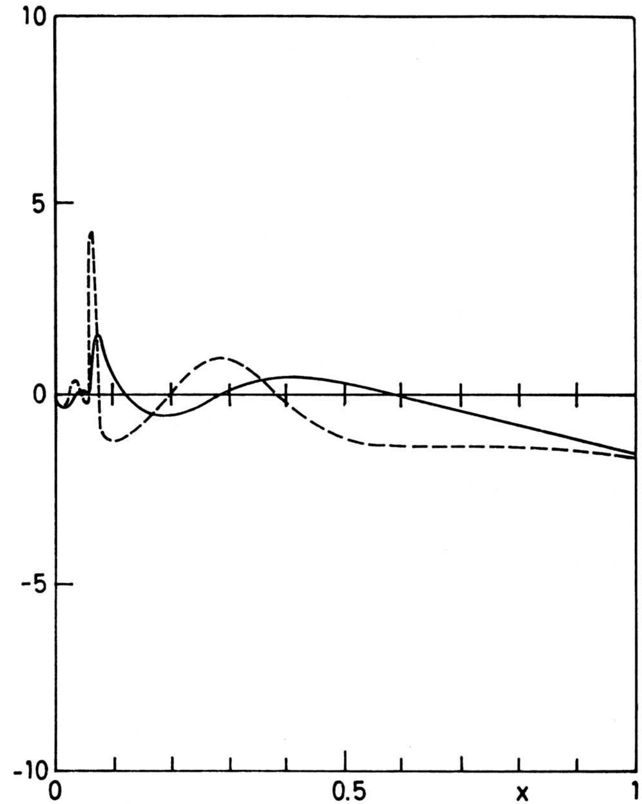

Fig. 7. — Same as Fig. 6, for mode g_6 (a mode of oscillation of the envelope).

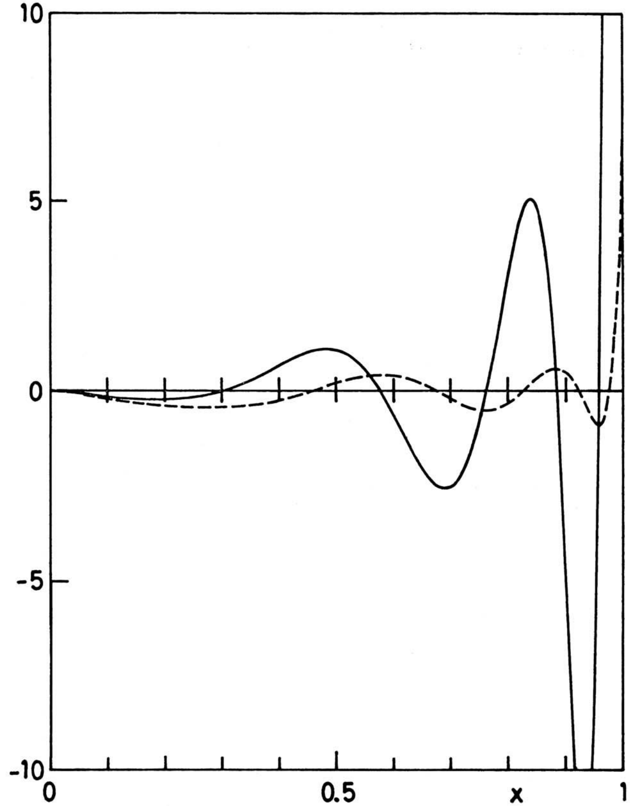

Fig. 9. — Same as Fig. 6, for mode p_5.

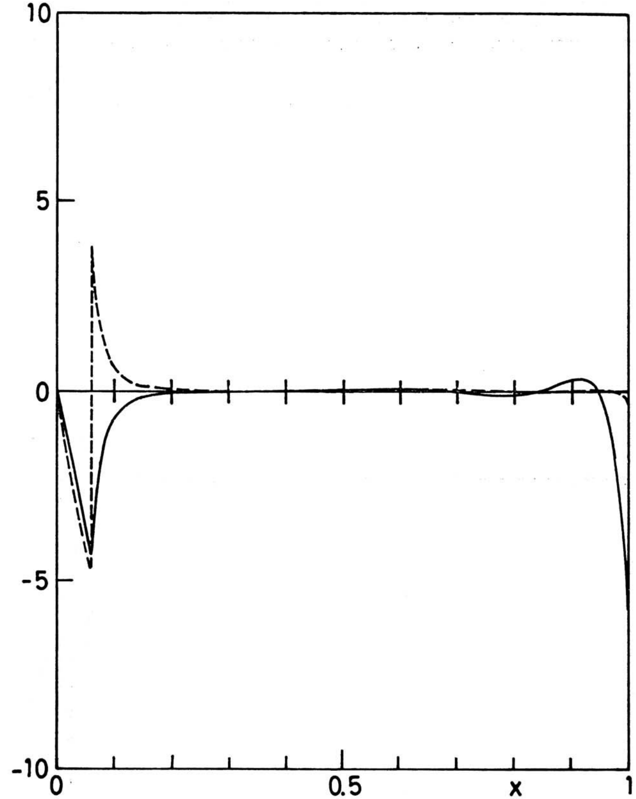

Fig. 8. — Same as Fig. 6, for mode p_4 (the discontinuity mode associated with the boundary of the core).

Another way to describe a non-radial mode rests on the construction of indices of the same nature as the mean and the standard deviation, the density of kinetic energy (which is a positive definite function) playing here the same role as the probability density does in statistics. If X is any quantity defined throughout the star we define a mean value of X :

$$\langle X \rangle = \int_0^1 X\, \xi^2 dq \,/ \int_0^1 \xi^2 dq.$$

In this way we define for a given mode a mean position $\langle x \rangle$. For simple models it comes out from the study of $\langle x \rangle$ the well-known fact that g modes concern mainly the central regions of the star, whereas p modes concern mainly the outer regions. Another very useful quantity is defined by

$$\Delta = 2 \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$$

It is analogous to a standard deviation. We have introduced the factor two so that its maximum value is one. For a perfectly trapped mode Δ would be zero.

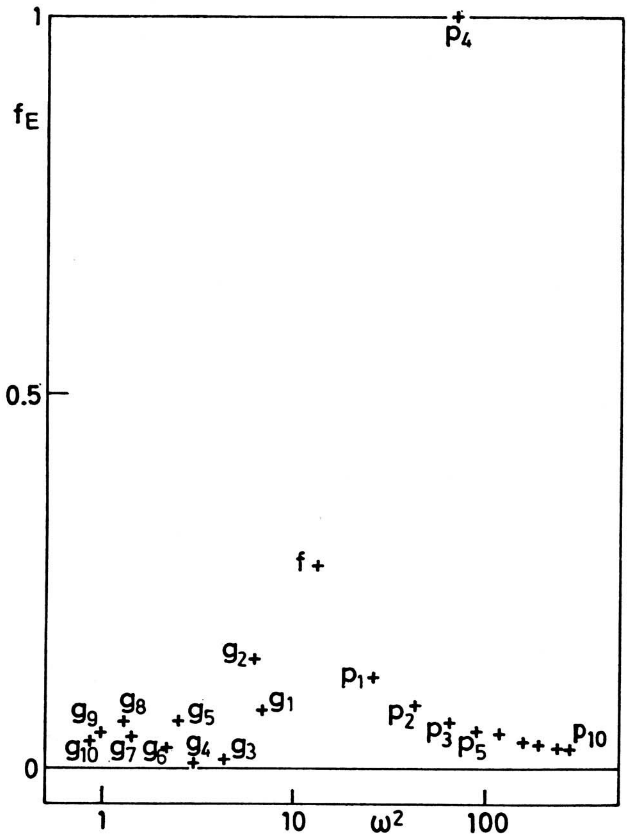

Fig. 10. — Fraction of the kinetic energy contained in the evanescent zone f_E versus ω^2 for some $l = 2$ modes of model 3.

Fig. 11 shows Δ for some $l = 2$ modes of model 3. The discontinuity mode p_4 (which is, in a certain sense, a limit case of a trapped mode) is clearly distinguished from neighbouring modes. Modes g_{10}, g_8, g_5 and g_2 are trapped in the core and are also characterized by a small value of Δ because of the small extent of the core (x_c = 0.0615). For trapped modes the value of $\langle x \rangle$ points out the region of trapping.

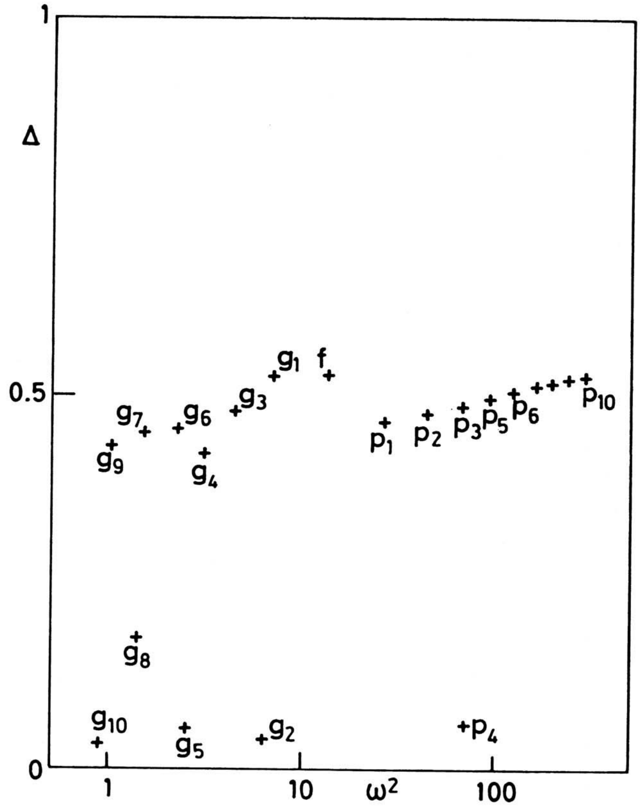

Fig. 11. — Values of Δ versus ω^2 for some $l = 2$ modes of model 3.

5. CONCLUSIONS

The different terms of the above decomposition of the energy have a clear physical meaning and may be used to describe physical characteristics of a non-radial mode. It is clear that the various indices we have constructed to this end cannot replace a visual inspection of the eigenfunctions. Nevertheless, when computing a large number of modes, we can use them for an automatic first sorting of the modes. Particularly, discontinuity modes and modes trapped in regions of small extent are well characterized by the index Δ.

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