PROPERTIES OF NONRADIAL STELLAR OSCILLATIONS

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ABSTRACT

It is shown that the oscillatory properties of the eigenfunctions can be proved rigorously for the second order problem. Models with discontinuities in density are also considered and "discontinuity modes" are shown to exist. The distribution of amplitudes of these modes is also discussed.

1. INTRODUCTION

The mathematical properties of the eigenvalue problem describing the adiabatic nonradial oscillations of stars are still poorly known and our information rests only on numerical integrations.

Even when the problem is simplified neglecting the Eulerian perturbation of the potential, a rigorous analysis of the eigenvalue problem is still lacking. As early as 1941, Cowling (1941) introduced the distinction between p and g spectra on the basis of an asymptotic discussion of the problem. Owen (1957) was unable to find the f mode and the first p and g modes for polytropes of high central condensation. Robe (1968) showed that these modes still exist but that they acquire extra nodes. Scuflaire (1974) and Osaki (1975) showed that a regularity can be found in all cases provided the nodes are counted in an appropriate way.

We show that a rigorous discussion of the oscillatory properties of the eigenvalue problem can be done when the Eulerian perturbation of the potential is neglected. Full demonstrations are given by Gabriel and Scuflaire (1979).

The discussion is also extended to cases where discontinuities in density are present in the star. For incompressible fluids, it is known that each density discontinuity gives rise to a new mode (called a discontinuity mode). We show that it is not always so in stars.

When a density discontinuity is present in the star, there can nevertheless exist one or more modes having their largest amplitudes in the vicinity of one of the discontinuities. This problem is discussed using a simplified model whose predictions allow the interpretation of numerical results obtained from physical models.

2. OSCILLATORY PROPERTIES OF NONRADIAL OSCILLATIONS

2.1. Equations and Boundary Conditions

The equations for nonradial oscillations neglecting the Eulerian perturbation of the gravitational potential (Cowling's approximation) are

$$\frac{dv}{dr} = a\,w \quad , \tag{1}$$

$$\frac{dw}{dr} = b\,v \quad , \tag{2}$$

where $v = f_1 r^2\,\delta r \quad , \qquad w = f_2\,\frac{p'}{\rho} \quad ,$

$$f_1 = \exp\left(\int_0^r \frac{1}{\Gamma_1}\frac{d\ln p}{dr}\,dr\right) \quad , \tag{3}$$

$$f_2 = \exp\left(\int_0^r A\,dr\right) \quad , \qquad A = \frac{d\ln\rho}{dr} - \frac{1}{\Gamma_1}\frac{d\ln p}{dr} \quad , \tag{4}$$

$$a = \left(\frac{\sigma_a^2}{\sigma^2} - 1\right)\frac{r^2}{c^2}\frac{f_1}{f_2} \quad , \qquad c^2 = \Gamma_1\,\frac{\Gamma}{\rho} \quad , \tag{5}$$

$$b = \frac{1}{r^2}\left(\sigma^2 - n^2\right)\frac{f_2}{f_1} \quad , \tag{6}$$

ℓ is the degree of surface spherical harmonic, c is the velocity of sound, $n = \sqrt{-Ag}$ is the Brunt-Väisälä frequency and $\sigma_a = [\ell(\ell+1)]^{1/2}\,c/r$ is the critical sound frequency.

Equations (1) and (2) are those given in Ledoux and Walraven (1958) modified to take the non-constancy of Γ_1 into account.

Equation (3) shows that f_1 is continuous throughout the star even when discontinuities in density are present. In such cases, A must be considered as a distribution to maintain the validity of equation (4); f_2 is discontinuous at discontinuities of density and satisfies the equation

$$\frac{f_{2+}}{\rho_+} = \frac{f_{2-}}{\rho_-} \quad , \tag{7}$$

where the subscripts - and + refer to the lower and upper sides of the discontinuity.

For what follows it is useful to represent the solutions in the [v(r), w(r)] plane (Scuflaire 1974) and to introduce the polar coordinates (ψ,θ) defined by

$$v = \psi \cos\theta \quad , \qquad w = \psi \sin\theta \quad . \tag{8}$$

Then equation (1) and (2) become

$$\frac{d\theta}{dr} = b \cos^2\theta - a \sin^2\theta \quad , \tag{9}$$

$$\frac{d\psi}{dr} = (a+b)\psi \sin\theta \cos\theta \quad .$$

The discussion of the properties of the eigenvalue problem is based on the behavior of the solutions of equation (9).

It is readily verified that the regularity condition at the center requires that v and w go to zero respectively as $r^{\ell+1}$ and r^{ℓ}, that $\lim_{r\to 0} \frac{rw}{v} = \frac{\sigma^2}{\ell}$ and $\theta(0,\sigma^2) = \pi/2 + k\pi$.

We may take k = 0 and we have

$$\theta(r,\sigma^2) = \frac{\pi}{2} - \frac{\ell}{\sigma^2} r$$

for sufficiently small r.

The boundary condition to apply at the "surface" is less obvious especially for non-zero surface temperature models. In all cases we are led to a condition of the form $\theta(R) = \alpha + k\pi$ with $0 < \alpha < \pi/2$.

For zero surface temperature models the boundary condition is $\delta p(R) = 0$, which is equivalent to the condition of regularity of the solution, implies that

$$\theta = tg^{-1}\left[\frac{GM}{R^4}\frac{f_2(R)}{f_1(R)}\right] + k\pi = \pi/2 + k\pi \quad .$$

For these models it can be considered that in the outermost layers ($r > r_2$) $m(r) = M$, $r \approx R$ and $P = K\rho^{\gamma}$ with γ constant. Then if $|n^2| \gg \sigma^2 \gg \sigma_a^2$ the regular solution is

$$\theta = k\pi + tg^{-1}(\beta x^{-m}) \quad , \tag{12}$$

with $m = \frac{2\gamma - \Gamma_1}{\Gamma_1(\gamma-1)}$, $\qquad x = R - r$, $\qquad$ and

$$\beta = \frac{m + \sqrt{m^2 - 4a_1b_1}}{2a_1} \,, \qquad a_1b_1 = \frac{\gamma(\gamma - \Gamma_1)}{\Gamma_1^2(\gamma-1)^2} \quad .$$

At a discontinuity in density δr and δp must be continuous. This implies the continuity of v and a discontinuity in w given by

$$w_+ - w_- = \frac{g}{r^2 f_1}\left(\frac{f_2}{\rho}\right)(\rho_+ - \rho_-)v \quad ,$$

and for θ

$$\mathrm{tg}\ \theta_+ - \mathrm{tg}\ \theta_- = \frac{g}{r^2 f_1}\left(\frac{f_2}{\rho}\right)(\rho_+ - \rho_-) \quad . \tag{13}$$

Obviously we may impose that $|\theta_+ - \theta_-| < \pi$ then θ_+ and θ_- belong to the same interval $I_k = (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$.

From the definition of a and b it is obvious that they are respectively decreasing and increasing functions of σ^2. Because of this simple property it is possible to demonstrate rigorously the oscillatory properties of the eigenfunctions of nonradial oscillations using exactly the same mathematical techniques as Coddington and Levinson (1955) for the Sturm-Liouville problem. The proofs are given in details by Gabriel and Scuflaire (1979).

If we assign to the nodes the sign of $\frac{d\theta}{dr}$ at that point, the properties are:

(1) All stars have a stable p spectrum of pressure modes, i.e., $\sigma_k^2 > 0$ $k = 1, 2, \ldots$, which has an accumulation point at infinity. The eigenfunction r_k associated to σ_k^2 has k zeros.

(2) There is a fundamental node associated to k = 0, with $\sigma_0^2 > 0$. The algebraic sum of the nodes of δr_0 is equal to zero.

(3) If the star has at least one radiative zone there is a stable g^+ spectrum of gravity modes, i.e., $\sigma_k^2 > 0$ $k = -1, -2, \ldots$, which has an accumulation point at zero. The eigenfunction δr_k associated to δ_k^2 has k zeros ($k < 0$).

(4) If the star has at least one convectively unstable zone there is an unstable g^- spectrum of gravity modes, sometimes called convective modes, i.e., $\sigma_k^2 < 0$, $k = +1, +2, \ldots$, which has an accumulation point at zero. The δr_k associated to σ_k^2 has k zeros. The smaller eigenvalue of σ_1^2 is larger than the minimum of the square of Brunt-Väisälä frequency.

(5) If there are N "unstable" discontinuities in the star such that the density below the discontinuity is smaller than above, then N new modes appear in the eigenvalue spectrum provided there is no convectively unstable zone, i.e., if there is no g^- spectrum. We call these modes unstable discontinuity modes. They are all unstable, i.e., $\sigma_j^2 < 0$ $j = 1, \ldots N$.

(6) If there are N "stable" discontinuities in the star such that the density below the discontinuity is larger than above, then N new modes appear in the eigenvalue spectrum provided there is no g^+ spectrum. (The homogeneous model is an example of such a situation.) We call these modes stable discontinuity

modes. They are all stable, i.e., $\sigma_1^2 > 0$ j = 1, . . . N.

3. BEHAVIOR OF EIGENFUNCTIONS NEAR A DISCONTINUITY

From the results of the preceding section we see that under some circumstances the presence of discontinuities in density introduces new modes, i.e., discontinuity modes, while under other circumstances, which are the most common ones, no new mode may be associated with the discontinuities. Nevertheless, it may be that some modes show large amplitudes near the discontinuity and may still be associated with it.

To check this we have searched for such modes in main sequence models of a 1.1 $M_\odot$ with X = 0.6 and Z = 0.044. For $X_c > 0.25$ these models have a small growing convective core on the top of which a "stable" discontinuity develops. As such models are not fully convective, there is no new mode associated to the discontinuity. A few properties of these models are given in Table 1.

Numerical calculations have been performed for values of the degree of spherical harmonic order ℓ equal to 10, 25, 50, and 100. Table 2 shows the results. It gives the eigenvalues ω^2 of the "discontinuity modes" (in unit GM/R^3), the ratio R_a of the value of $\delta r/r$ on the discontinuity to its largest value elsewhere in the star and the identification of the modes.

In all cases but one, only one discontinity mode was found. For model 1 and ℓ = 10, three modes with $R_a > 1$ were found. All other modes of these models show R_a values much smaller than one.

The existence of one discontinuity mode will not surprise but the presence, in one case, of three of them probably will. This can, however, be explained on the basis of a simple model.

For the amplitude to be large on the discontinuity it is necessary that the eigenfunctions have an exponential behavior on both sides of the discontinuity. Let us suppose that it is so and that the coefficients a and b in equations (1) and (2) may be considered as constant.

Below the discontinuity (subscript 1) the solution is given by

$$v_1 = e^{\lambda_1 (r-r_d)} \quad ,$$

$$w_1 = \sqrt{\frac{b_1}{a_1}} \; e^{\lambda_1 (r-r_d)} \quad ,$$

$$\text{with } \lambda = \sqrt{ab} \quad ,$$

where r_d is the radius on the discontinuity. We have taken v (r_d) = 1 and we have also dropped the other independent solution corresponding to the singular one at the

Table 1.

n°	$\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$	q	x	X_c
1	3,8 10^{-3}	3,41 10^{-2}	7,13 10^{-2}	0.568
2	2,3 10^{-2}	4,20 10^{-2}	7,15 10^{-2}	0.466
3	6,2 10^{-2}	4,83 10^{-2}	6,9 10^{-2}	0.332

A few properties of the models. q is the mass fraction and x the fractional radius at the discontinuity. X_c is the central hydrogen abundance.

TABLE 2 a

ℓ	10	25	50	100
ω^2	8.747	19,682	38.256	75,42
R_a	11,31	10^7	8 10^6	10^5
	g_6	g_4	f	f
ω^2	8,626			
R_a	78			
	g_7			
ω^2	7.478			
R_a	2,10			
	g_8			

Dimensionless eigenvalues ω^2 of discontinuity modes, ratio R of amplitudes on the discontinuity to the maximum value in the rest of the star and identification of the modes for model 1.

TABLE 2 b

ℓ	10	25	50	100
ω^2	51,38	120.1	233.8	454.2
R_a	4 10^6	2 10^{23}	4 10^{50}	4 10^{47}
	P_2	P_3	P_2	P_2

Same as Table 2 a but for model 2.

TABLE 2 c

ℓ	10	25	50	100
ω^2	145.0	340.5	66,2	1317
R_a	3 10^7	5 10^{22}	3 10^{49}	6 10^{44}
	P_5	P_6	P_6	P_6

Same as Table 2 a but for model 3.

center.

Above the discontinuity (subscript 2) we have

$$v_2 = A\, e^{\lambda_2(r-r_d)} + B\, e^{-\lambda_2(r-r_d)} \quad ,$$

$$w_2 = \sqrt{\frac{b_2}{a_2}}\left(A\, e^{\lambda_2(r-r_d)} + B\, e^{-\lambda_2(r-r_d)}\right) \quad .$$

The continuity of δr and δp give A and B and we have

$$v_2 = ch\left(\lambda_2(r-r_d)\right) + sh\left(\lambda_2(r-r_d)\right)\sqrt{\frac{a_2}{b_2}}\left\{\left[\frac{g}{r^2 f_1}\left(\frac{f_2}{\rho}\right)(\rho_+ - \rho_-)\right]_d + \sqrt{\frac{b_1}{a_1}}\right\} \quad ,$$

$$v_2 = ch\left(\lambda_2(r-r_d)\right) + sh\left(\lambda_2(r-r_d)\right)\left\{\left[\frac{\sigma_{a,2}^2 - \sigma^2}{\sigma_{a,1}^2 - \sigma^2}\;\frac{\sigma^2 - n_1^2}{\sigma^2 - n_2^2}\right]^{1/2}\frac{\rho_-}{\rho_+}\right.$$

$$\left. - \frac{\rho_+ + \rho_-}{\rho_+}\,\sigma_d^2\sqrt{\frac{\sigma^{-2} - \sigma_{a,2}^{-2}}{\sigma^2 - n_2^2}}\right\} \quad ,$$

$$\text{with } \sigma_d^2 = g\sqrt{\frac{\ell(\ell+1)}{r_d^2}}\;\frac{\rho_- - \rho_+}{\rho_- + \rho_+} \quad . \tag{14}$$

σ_d^2 is the eigenvalue of the discontinuity mode of an infinite incompressible fluid (see for instance Landau and Lifschitz 1969).

If for $r > r_x$ the eigenfunction oscillates, it can have its largest amplitude on the discontinuity only if $v_2(r_x) << 1$. This condition can be fulfilled only if σ^2 is in a narrow range around the value which gives $v_2(r_x) = 0$. Its general form is very complicated. More interesting is its expression when $\sigma_{a,1}^2 \simeq {}_{a,2}^2 >> \sigma^2 >> n_1^2$ n_2^2. Then $v(r_x) = 0$ if

$$\sigma^2 = \sigma_c^2 = \frac{\rho_+ + \rho_-}{\rho_+ \coth\left(\lambda_2(r_x - r_d)\right) + \rho_-}\,\sigma_d^2 \simeq \sigma_d^2$$

$$\text{if } \lambda_2(r_x - r_d) >> 1 \quad , \tag{15}$$

and $v(r_x) < 1$ if

$$\left|\frac{\sigma^2 - \sigma_c^2}{\sigma_c^2}\right| < \frac{\sigma_c^2}{\sigma_d^2} \frac{\rho_+}{\rho_+ + \rho_-} \left[sh\left(\lambda_2(r_x - r_d)\right)\right]^{-1} \tag{16}$$

$$\lesssim \frac{\rho_+}{\rho_+ \, ch\left(\lambda_2(r_x - r_d)\right) + \rho_- \, sh\left(\lambda_2(r_x - r_d)\right)} \approx \frac{2\rho_+ \, e^{-\lambda_2(r_x - r_d)}}{\rho_+ + \rho_-} \quad ,$$

if $\lambda_2(r_x - r_d) >> 1$.

If several eigenvalues satisfy condition (16) the corresponding modes are likely to have their largest amplitude on the discontinuity, i.e., to be discontinuity modes. This will occur the most easily if σ_c falls in the range of high order p or g modes.

The range of σ^2 defined by condition (16) decreases as $\lambda_2(r_x - r_d)$ increases. Since λ increases with ℓ for a given mode (in the models studied here we have $\lambda r_d \approx \ell$ in all cases), the probability to find several discontinuity modes decreases as ℓ or $(r_x - r_d)$ increases. For all usual models $(r_x - r_d)$ will be larger if σ_c^2 falls among p modes rather than among g^+ modes and more discontinuity modes are found in this later case. It is indeed when the discontinuity modes are g^+ modes of high order that we find several of them.

Since the behavior of the eigenfunction changes very rapidly when σ^2 deviates slightly from σ_c^2 it will always be possible to find at least one discontinuity mode, provided of course that the solutions do not oscillate on the other side of the discontinuity for $\sigma^2 \approx \sigma_c^2$. For this later case no discontinuity mode will be identified.

Even if equation (15) is a very crude one $\sigma_c^2 = \sigma_d^2$ gives a useful order of magnitude for the eigenvalue of discontinuity modes. In all cases we have met so far σ_d^2 predicts always the eigenvalues within less than a factor two.

The very peculiar amplitude distribution of discontinuity modes should be kept in mind when the stability of models with discontinuities is discussed.

REFERENCES

Coddington, E. and Levinson, N. 1955, Theory of Ordinary Differential Equations, (McGraw-Hill).

Cowling, T.C. 1941, Mon. Not. R. Astr. Soc., 101, 367.

Gabriel, M. and Scuflaire, R. 1979, Acta Astr., 29, (in press).

Landau, L. and Lifschitz, E. 1959, Fluid Mechanics, (Pergamon Press).

Ledoux, P. and Walraven, T. 1958, "Variable Stars" in Handbuch der Physik 51, 353, (ed. Flugge; Berlin: Springer).

Osaki, Y. 1975, Publ. Astron. Soc. Japan, 27, 237.

Owen, J.W. 1957, Mon. Not. R. Astr. Soc., 167, 384.

Robe, H. 1968, Ann. Astrophys., 31, 475.
Scuflaire, R. 1974, Astron. Astrophys., 36, 107.