PERIODIC ORBITS IN ANALYTICAL PLANAR GALACTIC POTENTIALS

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Abstract. We study the regular families of periodic orbits in an analytical planar galactic potential, using the method of Lindstedt. We obtain analytical expressions describing these orbits, validity of which is not limited to small amplitudes. We can delimit, in the space of the parameters, the domain of existence of each family of orbits.

Key words: galactic dynamics, periodic orbits

1. Introduction

It is well-known that the determination of periodic orbits is the first step in the study of a dynamical system. Such an investigation has been carried out by Davoust (1983) for a family of three-dimensional galactic potentials near resonances, using the method of Lindstedt. The periodic solutions of the equations of motion were formally written as power series of a parameter e (which is set equal to 1 at the end of the calculations). Setting equal to zero all terms which would give rise to secular terms, Davoust obtained equations of condition relating the phases and amplitudes of the components of motion and determined the solutions up to the first order in e. He was able to describe qualitatively the families of regular periodic orbits at and near resonances and his analytical solutions provided sufficiently accurate initial conditions to start an efficient numerical search.

Of course, the use of the Lindstedt's method is not limited to the first order, but the complexity of the algebra increases dramatically with the order of the computation. In a previous paper (Scuflaire, 1995, paper I in what follows) we used this method to study axial motions in a logarithmic potential. The use of an algebraic programming system (REDUCE) allowed us to obtain solutions in the form of power series of the amplitude a of the motion (total energy E can be used instead of a) up to high orders (typically between 15 and 20). The transformation into continued fractions gave rational expressions usable far outside the domain of convergence of the power series. In a second step, the study of the stability of axial orbits provided boundaries in the plane of the parameters for the loop and banana orbits resulting from the loss of the stability of axial orbits.

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The purpose of the present work is to extend the method to the direct determination of the regular two-dimensional periodic orbits up to high orders.

2. Equations of Motion

We use the two-dimensional logarithmic potential (Binney and Tremaine, 1987):

$$V(x, y) = \frac{1}{2} v_0^2 \ln \left(R_c^2 + x^2 + \frac{y^2}{q^2} \right), \tag{1}$$

where $q \leqslant 1$ defines the ellipticity of the equipotential curves, R_c is the core radius and v_0 the circular velocity at large distance from the center when $q = 1$. The motion in any other analytical potential regular at the origin would be studied in the same manner. Taking R_c as the unit of length and R_c/v_0 as the unit of time, the expression of the potential simplifies to

$$V(x, y) = \frac{1}{2} \ln \left(1 + x^2 + \frac{y^2}{q^2} \right). \tag{2}$$

The equations of motion read

$$\ddot{x} + \frac{x}{1 + x^2 + y^2/q^2} = 0, \tag{3}$$

$$\ddot{y} + \frac{y}{q^2(1 + x^2 + y^2/q^2)} = 0. \tag{4}$$

To understand the origin of different families of periodic orbits, it may be useful to recall that for infinitely small amplitude, the motion obeys the linear equations:

$$\ddot{x} + x = 0, \tag{5}$$

$$\ddot{y} + \frac{y}{q^2} = 0. \tag{6}$$

The angular frequencies of x and y components of motion are, respectively, $\sigma_x = 1$ and $\sigma_y = 1/q \geqslant \sigma_x$. The motion is periodic only if q is rational; $q = m/n$ where m and n are relatively prime with $m \leqslant n$. In this case the angular frequency of the movement is $\sigma = 2\pi/\text{period} = \sigma_x/m = \sigma_y/n = 1/m$. In a period, the particle performs m oscillations in the x direction and n oscillations in the y direction. We use the notation $m : n$ to describe the family of periodic orbits originating from this resonance. For motions of finite amplitude, the condition of existence of periodic orbits of type $m : n$ is not so demanding, q has only to belong to a neighbourhood of m/n, whose size increases with increasing energy E. In Section 5, we establish the precise limits of the domains of existence of different families of periodic solutions

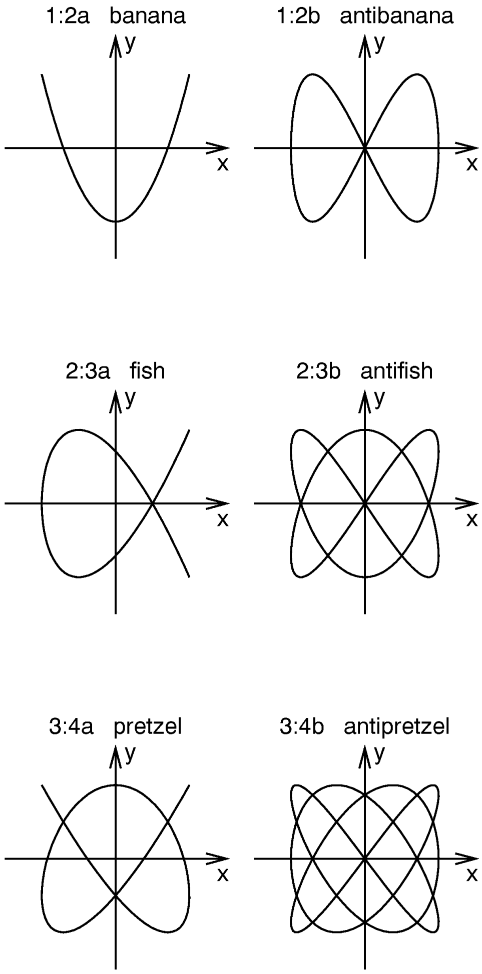

Figure 1. Schematic shapes of periodic orbits belonging to resonances 1 : 2, 2 : 3 and 3 : 4.

in the (q, E)-plane. For now, let us just say that the width Δq of the q-domain of existence of a periodic orbit of type $m : n$ is given by

$$\Delta q \approx \frac{m}{2n} E \tag{7}$$

for small values of the energy. A similar estimate ($\Delta q \propto E$) for a different potential has been obtained by Contopoulos (1965, Equation 137) where our q and E must be compared with his $\sqrt{A/B}$ and h.

The values of q lower than $1/2$ are of no interest for stellar dynamics, so in the following sections, we limit our study to the low order resonances $1:1$, $1:2$, $2:3$, $3:4$, $3:5$ and $4:5$. Figure 1 shows schematically the shapes of the periodic orbits belonging to a few of these resonances and the names given to the families of orbits parented by these periodic orbits (Miralda-Escudé and Schwarzschild, 1989).

3. Method of Solution

The use of the Lindstedt's method is well-known in celestial mechanics and described in a number of textbooks (see, e.g. Nayfeh, 1973 or Hayashi, 1985). It has been used and described by Presler and Broucke (1981a,b) and Davoust (1983) to determine periodic orbit families in galactic potentials. The equations of motion are rewritten as

$$[q^2 + \epsilon(q^2x^2 + y^2)]\ddot{x} + q^2x = 0, \tag{8}$$

$$[q^2 + \epsilon(q^2x^2 + y^2)]\ddot{y} + y = 0, \tag{9}$$

where a parameter ϵ has been inserted in front of the nonlinear terms. At the end of the calculation we will set $\epsilon = 1$. However, this parameter enables us to write the solutions as power series and to compute their terms iteratively:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots, \tag{10}$$

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots. \tag{11}$$

The angular frequency of the motion is also written as a power series of ϵ:

$$\sigma = \sigma_0 + \sigma_1\epsilon + \sigma_2\epsilon^2 + \cdots. \tag{12}$$

To simplify the notations, the variable $s = \sigma t$ will be used instead of t and the derivation with respect to s will be denoted by the prime symbol: $f' = \mathrm{d}f/\mathrm{d}s$.

At order 0, we recover the linear case, the solution corresponding to the $m : n$ resonance exists for $q = q_0 = m/n$ and is given by

$$x_0(s) = A\cos ms + B\sin ms, \tag{13}$$

$$y_0(s) = C\cos ns + D\sin ns, \tag{14}$$

with $\sigma = \sigma_0 = 1/m$. Since the system described by Equations (8) and (9) is autonomous, the substitution of t + constant for t in a solution gives a solution. This invariance can be used, by a proper 'choice of the time origin', to set to zero one of the coefficient or, in other words, to choose the phase of the x-component of motion, without loss of generality. We put

$$B = 0, \tag{15}$$

and we assume, in the following, that $A \neq 0$ (the motion along an axis has been studied in paper I). When proceeding to increasing orders, and imposing the vanishing of all secular terms, we obtain equations involving the constants A, C, D and the $x_k(t)$, $y_k(t)$ and σ_k for $k = 1, 2, \ldots$. However, these equations become rapidly inextricable.

Happily, the following trick enabled us to keep tractable equations. Instead of looking for a periodic solution for a given value of the parameter q, we determine q as a function of the coefficients A, C and D. In fact, only two of these coefficients are independent, as it is shown below. Let us denote them a and b. At the end of the computation, we obtain q and E (the energy) expressed in terms of a and b. These expressions can be inverted, numerically if necessary, to give a and b in terms of q and E.

As the other unknowns of the problem, q is written as a power series of ϵ:

$$q = q_0 + q_1\epsilon + q_2\epsilon^2 + \cdots . \tag{16}$$

4. Condition Equations

We know the order zero solution (Equations (13)–(15)). The solutions at increasing orders are obtained through a recurrence that we sketch below. Let us suppose that we have obtained a solution up to order $k-1$. The substitution of the power series in the equations of motion gives

$$x_k'' + m^2 x_k = 2m^3 x_0 \sigma_k + F_k, \tag{17}$$

$$y_k'' + n^2 y_k = \left(\frac{2n^3}{m} q_k + 2mn^2\sigma_k\right) y_0 + G_k, \tag{18}$$

where F_k and G_k are finite trigonometric expressions of s computed from the solution up to order $k-1$ already obtained. As we are seeking a periodic solution, we must avoid secular terms in x_k and y_k, that is, we must set to zero the coefficients of the terms in $\cos ms$ and $\sin ms$ in the right hand side of Equation (17) and the coefficients of the terms in $\cos ns$ and $\sin ns$ in the right hand side of Equation (18). These four algebraic equations express, at each order k, the conditions of periodicity of the solution and are called equations of condition. They are linear in σ_k and q_k but usually nonlinear in A, C and D. Generally, only two of these four conditions are independent so that they can be satisfied by a choice of the two parameters σ_k and q_k. When the equations of condition are satisfied, the solution of Equations (17) and (18) is unique except that arbitrary terms in $\cos ms$ and $\sin ms$ can be added to x_k and arbitrary terms in $\cos ns$ and $\sin ns$ can be added to y_k. The addition of these arbitrary terms would be equivalent to a redefinition of coefficients A, B, C and D introduced at order 0 and can thus be ignored. So we can consider that the solution of order $k \neq 0$ is unique and that x_k does not contain terms in $\cos ms$ and $\sin ms$ and that y_k does not contain terms in $\cos ns$ and $\sin ns$.

However, at order $m+n-1$, the equations of condition for periodic motion cannot be solved with respect to σ_k and q_k for arbitrary values of A, C and D. The condition of solvability imposes a relation between the coefficients C and D. In other terms, it imposes the phase of the y-component of motion. As we have already chosen the phase of the x-component (Equation (15)), we can also say that this relation locks the difference of phases between the two components of the motion. Similar phase lockings are described by Presler and Broucke (1981a) and Davoust (1983). This relation between C and D is given for each studied resonance case in the following section. As an example, we give the details of the calculation leading to this relation in the case of resonance 1 : 2, where it appears at order 2. In all studied cases, the detailed calculation shows that this relation between C and D occurs at order $m + n - 1$, but we are not able to explain in a simple and general way why this relation appears at that particular order. For each resonance, this equation admits several solutions corresponding to different families. These families depend on two arbitrary constants, a and b, and are described in the following section.

5. Periodic Orbit Families

The series we have computed were truncated at an order depending on the family, imposed by the limited memory available on the computer (of the order of 100 MB). As it is well-known, the intermediate computations require much more computer resource than the final result. For each family, the motion can be described by a computer file, the size of which is of the order of 1 MB or slightly greater. For families belonging to resonances 1 : 1, 1 : 2, 2 : 3 and 3 : 4, the series were computed up to order 15, whereas for families belonging to resonances 3 : 5 and 4 : 5 we were limited to order 10 (and even 9 for subfamily 4 : 5b). The series cannot be used directly for numerical computations because they converge slower and slower as the energy is increased and finally diverge. As in paper I, the transformation of the power series into continued fractions allows us to extend significantly the domain of convergence of our results. The program computing the coefficients of the continued fractions from the coefficients of the power series has been written in FORTRAN. We have used the analytical results to determine the domains of existence of different families of periodic orbits (see below). These results were confirmed numerically, as is explained at the end of this section.

5.1. RESONANCE 1 : 1

The condition of solvability of the condition equations at order 1 imposes the following relation between C and D:

$$CD = 0. \tag{19}$$

This gives two families that we label 1 : 1a and 1 : 1b. Of course, we can only give the first few terms of the power series in the present paper (the files describing the full series can be obtained upon request).

5.1.1. *Family* $1:1a$
With $A = a$, $C = b$ and $D = 0$, we obtain

$$
\begin{aligned}
q &= 1, & (20)\\
\sigma &= 1 - \frac{3\epsilon}{8}(a^2+b^2) + \frac{65\epsilon^2}{256}(a^2+b^2)^2 + \cdots, & (21)\\
E &= \frac{1}{2}(a^2+b^2) - \frac{9\epsilon}{32}(a^2+b^2)^2 + \frac{1355\epsilon^2}{6144}(a^2+b^2)^3 + \cdots, & (22)\\
\frac{x}{a} &= \frac{y}{b} = \cos s - \frac{\epsilon}{32}(a^2+b^2)\cos 3s + \\
&\quad + \frac{\epsilon^2}{3072}(a^2+b^2)^2(57\cos 3s + 11\cos 5s) + \cdots . & (23)
\end{aligned}
$$

Family $1:1a$ exists only for the particular case of an axisymmetric potential ($q = 1$) and its orbits are rectilinear.

5.1.2. *Family* $1:1b$
With $A = a$, $C = 0$ and $D = b$, we obtain

$$
\begin{aligned}
q &= 1 + \frac{\epsilon}{4}(a^2-b^2) + \frac{\epsilon^2}{64}(-7a^4 + 4a^2b^2 + 3b^4) + \cdots, & (24)\\
\sigma &= 1 + \frac{\epsilon}{8}(-3a^2-b^2) + \frac{\epsilon^2}{256}(65a^4 + 34a^2b^2 - 3b^4) + \cdots, & (25)\\
E &= \frac{1}{2}(a^2+b^2) + \frac{\epsilon}{32}(-9a^4 - 14a^2b^2 - b^4) + \\
&\quad + \frac{\epsilon^2}{6144}(1355a^6 + 2517a^4b^2 + 309a^2b^4 - 85b^6) + \cdots, & (26)\\
x &= a\cos s + \frac{\epsilon}{32}(-a^3 + ab^2)\cos 3s + \\
&\quad + \frac{\epsilon^2}{3072}[(57a^5 - 54a^3b^2 - 3ab^4)\cos 3s + \\
&\quad + (11a^5 - 22a^3b^2 + 11ab^4)\cos 5s] + \cdots, & (27)\\
y &= b\sin s + \frac{\epsilon}{32}(-a^2b + b^3)\sin 3s + \\
&\quad + \frac{\epsilon^2}{3072}[(51a^4b - 42a^2b^3 - 9b^5)\sin 3s + \\
&\quad + (11a^4b - 22a^2b^3 + 11b^5)\sin 5s] + \cdots . & (28)
\end{aligned}
$$

The members of this family are loop orbits. A change in the sign of a or b reverses the direction of motion on the same orbit. The limits of the domain of existence of this family are obtained by setting $a = 0$ or $b = 0$ in the above expressions. When $a = 0$, periodic loop orbits degenerate into y-axial orbits. In the (q, E)-plane these

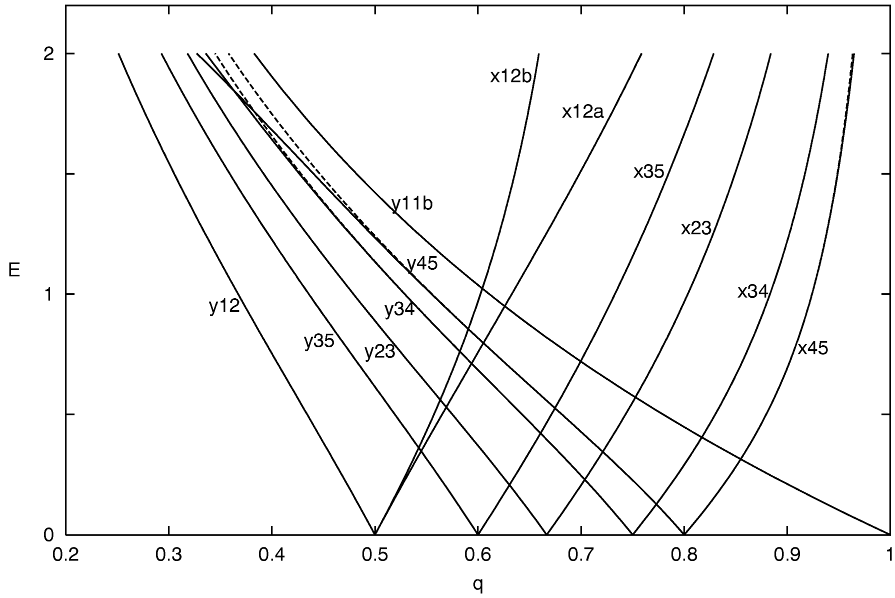

Figure 2. Limits of existence of the families of periodic orbits computed analytically (solid line) and numerically (dashed line).

orbits lie along the curve labeled $y11b$ in Figure 2, given by

$$q = 1 - \frac{\epsilon}{4}b^2 + \frac{3\epsilon^2}{64}b^4 + \cdots , \tag{29}$$

$$E = \frac{1}{2}b^2 - \frac{\epsilon}{32}b^4 - \frac{85\epsilon^2}{6144}b^6 + \cdots . \tag{30}$$

The elimination of b between these equations gives

$$q = 1 - \frac{\epsilon}{2}E + \frac{\epsilon^2}{8}E^2 + \cdots . \tag{31}$$

This curve is also the limit of stability of the y-axial orbits, as shown in paper I. To compare with the results given in this paper, the following remarks must be taken into account. Figure 6(a) of paper I uses the amplitude instead of the energy. The expression of q_{b1} given by Equation (59) relates to an x-orbit; for a y-orbit, the appropriate expression is $1/q_{b1}$. When ϵ, the square of the amplitude, in these expressions is substituted for the energy (Equation (55) of paper I), we recover the expression given by Equation (31).

If we set $b = 0$, we obtain orbits degenerated along the x-axis, with values of $q > 1$.

$$q' = 1 + \frac{\epsilon}{2}E + \frac{\epsilon^2}{8}E^2 + \cdots \tag{32}$$

q' describes the same limit as q when we exchange the roles of x and y axes. Consequently $q' = 1/q$; q' can be compared more easily to q_{b1} of paper I.

5.2. RESONANCE 1:2

As an example, we give below a detailed exposition of the application of the Lindstedt's method to the resonance 1:2, up to second order, where the equations of condition impose a relation between the coefficients C and D.

As explained in Section 3, at order 0, the solution is given by

$$\sigma_0 = 1, \tag{33}$$

$$q_0 = \tfrac{1}{2}, \tag{34}$$

$$x_0 = A\cos s, \tag{35}$$

$$y_0 = C\cos 2s + D\sin 2s. \tag{36}$$

Now, in Equations (8) and (9), we substitute the expressions given by Equations (10)–(12) and (16) for $x(t)$, $y(t)$, σ and q. At first order in ϵ, we obtain

$$\begin{aligned} x_1'' + x_1 &= \tfrac{1}{4}(8A\sigma_1 + 8AD^2 + 8AC^2 + 3A^3)\cos s + \\ &\quad + \tfrac{1}{4}(-4AD^2 + 4AC^2 + A^3)\cos 3s + 2ACD\sin 3s + \\ &\quad + (AC^2 - AD^2)\cos 5s + 2ACD\sin 5s, \end{aligned} \tag{37}$$

$$\begin{aligned} y_1'' + 4y_1 &= A^2C + (8C\sigma_1 + 16Cq_1 + 12CD^2 + 12C^3 + 2A^2C)\cos 2s + \\ &\quad + (8D\sigma_1 + 16Dq_1 + 12D^3 + 12C^2D + 2A^2D)\sin 2s + \\ &\quad + A^2C\cos 4s + A^2D\sin 4s + (-12CD^2 + 4C^3)\cos 6s + \\ &\quad + (-4D^3 + 12C^2D)\sin 6s. \end{aligned} \tag{38}$$

In order to avoid secular terms in the first order solution, we must set to zero the coefficients of $\cos s$ and $\sin s$ in the right hand member of the first equation and the coefficients of $\cos 2s$ and $\sin 2s$ in the right hand member of the second equation. The coefficient of $\sin s$ is already zero and we are left with three equations of condition.

$$8A\sigma_1 + 8AD^2 + 8AC^2 + 3A^3 = 0, \tag{39}$$

$$8C\sigma_1 + 16Cq_1 + 12CD^2 + 12C^3 + 2A^2C = 0, \tag{40}$$

$$8D\sigma_1 + 16Dq_1 + 12D^3 + 12C^2D + 2A^2D = 0. \tag{41}$$

Two of these equations are independent and give us σ_1 and q_1:

$$\sigma_1 = -\tfrac{3}{8}A^2 - C^2 - D^2, \tag{42}$$

$$q_1 = \tfrac{1}{16}A^2 - \tfrac{1}{4}C^2 - \tfrac{1}{4}D^2. \tag{43}$$

Equations (37) and (38) now admit the periodic solution

$$\begin{aligned} x_1 &= \tfrac{1}{32}(4AD^2 - 4AC^2 - A^3)\cos 3s - \tfrac{1}{4}ACD\sin 3s + \\ &\quad + \tfrac{1}{24}(AD^2 - AC^2)\cos 5s - \tfrac{1}{12}ACD\sin 5s, \end{aligned} \tag{44}$$

$$\begin{aligned} y_1 &= \tfrac{1}{4}A^2C - \tfrac{1}{12}A^2C\cos 4s - \tfrac{1}{12}A^2D\sin 4s + \\ &\quad + \tfrac{1}{8}(3CD^2 - C^3)\cos 6s + \tfrac{1}{8}(D^3 - 3C^2D)\sin 6s. \end{aligned} \tag{45}$$

At order 2, the same process gives us the following differential equations:

$$\begin{aligned} x_2'' + x_2 &= \tfrac{1}{384}(768A\sigma_2 - 1216AD^4 - 2432AC^2D^2 - 1216AC^4 - \\ &\quad - 880A^3D^2 - 976A^3C^2 - 195A^5)\cos s - \tfrac{1}{4}A^3CD\sin s + \\ &\quad + \tfrac{1}{384}(480AD^4 - 480AC^4 + 176A^3D^2 - \\ &\quad - 304A^3C^2 - 57A^5)\cos 3s + \\ &\quad + \tfrac{1}{4}(-10ACD^3 - 10AC^3D - 5A^3CD)\sin 3s + \\ &\quad + \tfrac{1}{384}(480AD^4 - 480AC^4 + 496A^3D^2 - \\ &\quad - 496A^3C^2 - 33A^5)\cos 5s + \\ &\quad + \tfrac{1}{12}(-30ACD^3 - 30AC^3D - 31A^3CD)\sin 5s + \\ &\quad + \tfrac{1}{48}(-66AD^4 + 396AC^2D^2 - 66AC^4 + \\ &\quad + 35A^3D^2 - 35A^3C^2)\cos 7s + \\ &\quad + \tfrac{1}{24}(132ACD^3 - 132AC^3D - 35A^3CD)\sin 7s + \\ &\quad + \tfrac{1}{24}(-31AD^4 + 186AC^2D^2 - 31AC^4)\cos 9s + \\ &\quad + \tfrac{1}{6}(31ACD^3 - 31AC^3D)\sin 9s, \end{aligned} \tag{46}$$

$$\begin{aligned} y_2'' + 4y_2 &= \tfrac{1}{16}(-20A^2CD^2 - 20A^2C^3 - 9A^4C) + \tfrac{1}{48}(384C\sigma_2 + \\ &\quad + 768Cq_2 - 792CD^4 - 1584C^3D^2 - 792C^5 - \\ &\quad - 568A^2CD^2 - 568A^2C^3 - 73A^4C)\cos 2s + \\ &\quad + \tfrac{1}{48}(384D\sigma_2 + 768Dq_2 - 792D^5 - 1584C^2D^3 - \\ &\quad - 792C^4D - 568A^2D^3 - 568A^2C^2D - 67A^4D)\sin 2s + \\ &\quad + \tfrac{1}{48}(-142A^2CD^2 - 22A^2C^3 - 27A^4C)\cos 4s + \\ &\quad + \tfrac{1}{48}(-82A^2D^3 + 38A^2C^2D - 27A^4D)\sin 4s + \\ &\quad + \tfrac{1}{48}(792CD^4 + 528C^3D^2 - 264C^5 + 480A^2CD^2 - \\ &\quad - 160A^2C^3 - 19A^4C)\cos 6s + \tfrac{1}{48}(264D^5 - 528C^2D^3 - \\ &\quad - 792C^4D + 160A^2D^3 - 480A^2C^2D - 19A^4D)\sin 6s + \\ &\quad + \tfrac{1}{24}(231A^2CD^2 - 77A^2C^3)\cos 8s + \\ &\quad + \tfrac{1}{24}(77A^2D^3 - 231A^2C^2D)\sin 8s + \\ &\quad + \tfrac{1}{2}(-55CD^4 + 110C^3D^2 - 11C^5)\cos 10s + \\ &\quad + \tfrac{1}{2}(-11D^5 + 110C^2D^3 - 55C^4D)\sin 10s. \end{aligned} \tag{47}$$

The equations of condition read

$$768A\sigma_2 - 1216AD^4 - 2432AC^2D^2 - 1216AC^4 - \\ - 880A^3D^2 - 976A^3C^2 - 195A^5 = 0, \tag{48}$$

$$A^3CD = 0, \tag{49}$$

$$384C\sigma_2 + 768Cq_2 - 792CD^4 - 1584C^3D^2 - 792C^5 - \\ - 568A^2CD^2 - 568A^2C^3 - 73A^4C = 0, \tag{50}$$

$$384D\sigma_2 + 768Dq_2 - 792D^5 - 1584C^2D^3 - 792C^4D - \\ - 568A^2D^3 - 568A^2C^2D - 67A^4D = 0. \tag{51}$$

These condition equations cannot be solved with respect to σ_2 and q_2 for an arbitrary choice of the coefficients A, C and D. As we exclude the possibility $A = 0$ (axial movements, already studied in paper I), the condition of solvability reads

$$CD = 0. \tag{52}$$

This gives two families that we label $1:2a$ and $1:2b$.

5.2.1. *Family* $1:2a$

With $C = b$ and $D = 0$, we obtain

$$q = \frac{1}{2} + \frac{\epsilon}{16}(a^2 - 4b^2) + \frac{\epsilon^2}{1536}(-49a^4 + 160a^2b^2 + 368b^4) + \cdots, \tag{53}$$

$$\sigma = 1 + \frac{\epsilon}{8}(-3a^2 - 8b^2) + \frac{\epsilon^2}{768}(195a^4 + 976a^2b^2 + 1216b^4) + \cdots, \tag{54}$$

$$E = \frac{1}{2}(a^2 + 4b^2) + \frac{\epsilon}{32}(-9a^4 - 64a^2b^2 - 80b^4) + \frac{\epsilon^2}{18432}(4065a^6 + \\ + 40744a^4b^2 + 100000a^2b^4 + 86592b^6) + \cdots, \tag{55}$$

$$x = a\cos s + \frac{\epsilon}{96}[(-3a^3 - 12ab^2)\cos 3s - 4ab^2\cos 5s] + \\ + \frac{\epsilon^2}{46080}[(855a^5 + 4560a^3b^2 + 7200ab^4)\cos 3s + \\ + (165a^5 + 2480a^3b^2 + 2400ab^4)\cos 5s + \\ + (700a^3b^2 + 1320ab^4)\cos 7s + 744ab^4\cos 9s] + \cdots, \tag{56}$$

$$y = b\cos 2s + \frac{\epsilon}{24}[6a^2b - 2a^2b\cos 4s - 3b^3\cos 6s] + \\ + \frac{\epsilon^2}{23040}[-3240a^4b - 7200a^2b^3 + (1080a^4b + 880a^2b^3)\cos 4s + \\ + (285a^4b + 2400a^2b^3 + 3960b^5)\cos 6s + \\ + 1232a^2b^3\cos 8s + 1320b^5\cos 10s] + \cdots. \tag{57}$$

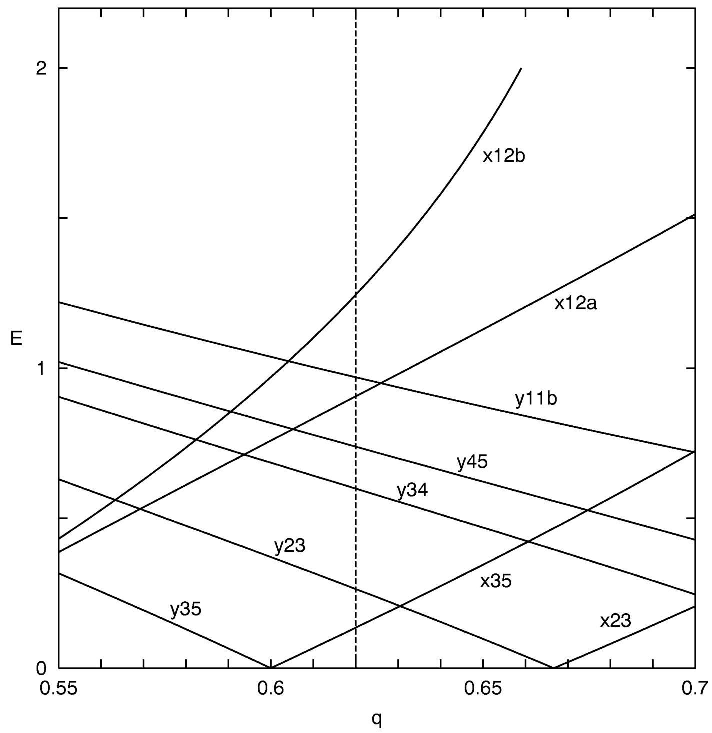

Figure 3. Limits of existence of families of periodic orbits in the neighbourhood of $q = 0.62$.

These equations describe periodic banana orbits. A change of sign of b gives an orbit which is symmetric to the original one with respect to the x-axis, whereas a change of sign of a leaves it invariant. The limits of the domain of existence of the family are obtained by setting $a = 0$ or $b = 0$ in the above expressions. Proceeding in the same manner as for the loop orbits, we obtain, in the (q, E)-plane the boundary where banana orbits degenerate into x-axial orbits (curve $x12a$ in Figure 2)

$$q = \frac{1}{2} + \frac{\epsilon}{8}E + \frac{5\epsilon^2}{384}E^2 + \cdots \tag{58}$$

and the boundary where they degenerate into y-axial orbits (curve $y12$ in Figure 2)

$$q = \frac{1}{2} - \frac{\epsilon}{8}E - \frac{7\epsilon^2}{384}E^2 + \cdots \tag{59}$$

Curve $x12a$ is also a limit of stability for x-axial movement, as it is shown in paper I (curve $a2$ of Figures 3(a) and 6(a)). When expressed in terms of E, the expression of q_{a2} (Equation (60) of paper I) reduces to our Equation (58).

5.2.2. *Family* $1:2b$

With $C = 0$ and $D = b$, we obtain

$$q = \frac{1}{2} + \frac{\epsilon}{16}(a^2 - 4b^2) + \frac{\epsilon^2}{1536}(-61a^4 + 256a^2b^2 + 368b^4) + \cdots, \quad (60)$$

$$\sigma = 1 + \frac{\epsilon}{8}(-3a^2 - 8b^2) + \frac{\epsilon^2}{768}(195a^4 + 880a^2b^2 + 1216b^4) + \cdots, \quad (61)$$

$$\begin{aligned} E = {} & \frac{1}{2}(a^2 + 4b^2) + \frac{\epsilon}{32}(-9a^4 - 64a^2b^2 - 80b^4) + \frac{\epsilon^2}{18432}(4065a^6 + \\ & + 38296a^4b^2 + 90784a^2b^4 + 86592b^6) + \cdots, \end{aligned} \quad (62)$$

$$\begin{aligned} x = {} & a\cos s + \frac{\epsilon}{96}[(-3a^3 + 12ab^2)\cos 3s + 4ab^2\cos 5s] + \\ & + \frac{\epsilon^2}{46080}[(855a^5 - 2640a^3b^2 - 7200ab^4)\cos 3s + \\ & + (165a^5 - 2480a^3b^2 - 2400ab^4)\cos 5s + \\ & + (-700a^3b^2 + 1320ab^4)\cos 7s + 744ab^4\cos 9s] + \cdots, \end{aligned} \quad (63)$$

$$\begin{aligned} y = {} & b\sin 2s + \frac{\epsilon}{24}(-2a^2b\sin 4s + 3b^3\sin 6s) + \frac{\epsilon^2}{23040}[(1080a^4b + \\ & + 3280a^2b^3)\sin 4s + (285a^4b - 2400a^2b^3 - 3960b^5)\sin 6s - \\ & - 1232a^2b^3\sin 8s + 1320b^5\sin 10s] + \cdots. \end{aligned} \quad (64)$$

These equations describe periodic antibanana orbits. A change of sign of b reverses the direction of motion on the same trajectory, whereas a change of sign of a leaves it invariant. The limits of the domain of existence of the family are obtained by setting $a = 0$ or $b = 0$ in the above expressions. In the (q, E)-plane the boundary where antibanana orbits degenerate into x-axial orbits (curve $x12b$ in Figure 2) is given by

$$q = \frac{1}{2} + \frac{\epsilon}{8}E - \frac{7\epsilon^2}{384}E^2 + \cdots. \quad (65)$$

They degenerate into y-axial orbits along the same boundary as the orbits of family $1:2a$ (curve $y12$ in Figure 2).

Curve $x12b$ is also a limit of stability for x-axial movement, as it is shown in paper I (curve $b2$ of Figures 3(a) and 6(a)). When expressed in terms of E, the expression of q_{b2} (Equation (61) of paper I) reduces to our Equation (65). For a given value of q, curves $x12a$ and $x12b$ delimit a range of energy in which x-axial movements are unstable.

5.3. RESONANCE $2:3$

To solve the equations of condition at order 4, we must impose the relation

$$CD(D^2 - C^2) = 0. \quad (66)$$

At first sight, this equation has four families of solutions:

$(a1)$ $C \neq 0$, $D = 0$,
$(a2)$ $C = 0$, $D \neq 0$,
$(b1)$ $D = C$,
$(b2)$ $D = -C$.

In fact, it appears that the change of variables

$$s = s' + \tfrac{1}{2}\pi \tag{67}$$

transforms a solution of type $(a1)$ into a solution of type $(a2)$ so that there is no need to distinguish between the two types. The same remark holds for the types $(b1)$ and $(b2)$. So, we have only two families that we label $2:3a$ and $2:3b$.

5.3.1. *Family* $2:3a$
With $C = b$ and $D = 0$, we obtain

$$\begin{aligned} q &= \frac{2}{3} + \frac{\epsilon}{48}(4a^2 - 9b^2) + \\ &+ \frac{\epsilon^2}{30720}(-1616a^4 + 4464a^2b^2 + 2349b^4) + \cdots , \end{aligned} \tag{68}$$

$$\begin{aligned} \sigma &= \frac{1}{2} + \frac{\epsilon}{32}(-6a^2 - 9b^2) + \\ &+ \frac{\epsilon^2}{5120}(650a^4 + 1656a^2b^2 + 1377b^4) + \cdots , \end{aligned} \tag{69}$$

$$\begin{aligned} E &= \frac{1}{8}(4a^2 + 9b^2) + \frac{\epsilon}{512}(-144a^4 - 576a^2b^2 - 405b^4) + \\ &+ \frac{\epsilon^2}{9830400}(2168000a^6 + 11675232a^4b^2 + \\ &+ 15887232a^2b^4 + 9035955b^6) + \cdots , \end{aligned} \tag{70}$$

$$\begin{aligned} x &= a\cos 2s + \frac{\epsilon}{160}(-30ab^2\cos 4s - 5a^3\cos 6s - 6ab^2\cos 8s) + \\ &+ \frac{\epsilon^2}{8601600}[-1058400a^3b^2 + (1189440a^3b^2 + \\ &+ 1134000ab^4)\cos 4s + (159600a^5 + 30240a^3b^2)\cos 6s + \\ &+ (455616a^3b^2 + 226800ab^4)\cos 8s + \\ &+ (30800a^5 + 179550ab^4)\cos 10s + 95904a^3b^2\cos 12s + \\ &+ 74655ab^4\cos 14s] + \cdots , \end{aligned} \tag{71}$$

$$
\begin{aligned}
y \;=\;& b\cos 3s + \frac{\epsilon}{640}(180a^2 b\cos s - 36a^2 b\cos 7s - 45b^3\cos 9s) \;+\\
&+ \frac{\epsilon^2}{2867200}[(-453600a^4 b - 438480a^2 b^3)\cos s \;+\\
&+ (12600a^4 b - 132300a^2 b^3)\cos 5s \;+\\
&+ (90720a^4 b + 39312a^2 b^3)\cos 7s \;+\\
&+ (191520a^2 b^3 + 155925b^5)\cos 9s + 21240a^4 b\cos 11s \;+\\
&+ 66528a^2 b^3\cos 13s + 51975b^5\cos 15s] + \cdots . \qquad (72)
\end{aligned}
$$

These equations describe periodic fish orbits. A change of sign of a gives an orbit symmetric to the original one with respect to the y-axis, whereas a change of sign of b leaves it invariant. The limits of the domain of existence of the family are obtained by setting $a = 0$ or $b = 0$ in the above expressions. Proceeding as above, we obtain, in the (q, E)-plane the boundary where fish orbits degenerate into x-axial orbits (curve $x23$ in Figure 2)

$$
q = \frac{2}{3} + \frac{\epsilon}{6}E - \frac{11\epsilon^2}{480}E^2 + \cdots \qquad (73)
$$

and the boundary where they degenerate into y-axial orbits (curve $y23$ in Figure 2)

$$
q = \frac{2}{3} - \frac{\epsilon}{6}E - \frac{7\epsilon^2}{160}E^2 + \cdots . \qquad (74)
$$

5.3.2. *Family* $2\!:\!3b$

With $C = D = b/\sqrt{2}$, we obtain expressions of q, σ and E which coincide with the corresponding expressions for family $2\!:\!3a$ up to order 3; x and y are given by

$$
\begin{aligned}
x \;=\;& a\cos 2s + \frac{\epsilon}{160}(-30ab^2\sin 4s - 5a^3\cos 6s - 6ab^2\sin 8s) \;+\\
&+ \frac{\epsilon^2}{8601600}[(1189440a^3 b^2 + 1134000ab^4)\sin 4s \;+\\
&+ (159600a^5 + 30240a^3 b^2)\cos 6s + (455616a^3 b^2 + 226800ab^4)\sin 8s \;+\\
&+ (30800a^5 - 179550ab^4)\cos 10s + 95904a^3 b^2\sin 12s \;-\\
&- 74655ab^4\cos 14s] + \cdots , \qquad (75)
\end{aligned}
$$

$$
\begin{aligned}
y \;=\;& \frac{b}{\sqrt{2}}(\cos 3s + \sin 3s) + \frac{\epsilon}{640\sqrt{2}}[180a^2 b(\cos s - \sin s) - 36a^2 b(\cos 7s \;+\\
&+ \sin 7s) + 45b^3(\cos 9s - \sin 9s) + \frac{\epsilon^2}{2867200\sqrt{2}}[(-453600a^4 b \;-
\end{aligned}
$$

$$- 438480a^2b^3)(\cos s - \sin s) + (12600a^4b + 132300a^2b^3)(\cos 5s -$$
$$- \sin 5s) + (90720a^4b + 39312a^2b^3)(\cos 7s + \sin 7s) +$$
$$+ (-191520a^2b^3 - 155925b^5)(\cos 9s - \sin 9s) +$$
$$+ 21240a^4b(\cos 11s + \sin 11s) - 66528a^2b^3(\cos 13s - \sin 13s) -$$
$$- 51975b^5(\cos 15s + \sin 15s)] + \cdots . \tag{76}$$

These equations describe periodic antifish orbits. A change of sign of a reverses the direction of motion on the same trajectory, whereas a change of sign of b leaves it invariant. In the (q, E)-plane, they have the same domain of existence as fish orbits and degenerate into axial orbits on the same boundaries.

5.4. RESONANCE 3:4

At order 6, we obtain the relation

$$CD(C^2 - 3D^2)(3C^2 - D^2) = 0. \tag{77}$$

This equation admits six types of solutions:

$(a1)$ $D = 0$,
$(a2)$ $D = -\sqrt{3}C$,
$(a3)$ $D = \sqrt{3}C$,
$(b1)$ $C = 0$,
$(a2)$ $C = \sqrt{3}D$,
$(a3)$ $C = -\sqrt{3}D$.

The transformations

$$s = s' + \tfrac{1}{3}\pi \quad \text{and} \quad s = s'' + \tfrac{2}{3}\pi \tag{78}$$

allow us to group them into two families.

5.4.1. *Family* 3:4*a*

With $C = b$ and $D = 0$, we obtain

$$q = \frac{3}{4} + \frac{\epsilon}{96}(9a^2 - 16b^2) + $$
$$+ \frac{\epsilon^2}{7168}(-465a^4 + 1216a^2b^2 + 256b^4) + \cdots , \tag{79}$$

$$\sigma = \frac{1}{3} + \frac{\epsilon}{216}(-27a^2 - 32b^2) + $$
$$+ \frac{\epsilon^2}{145152}(12285a^4 + 23424a^2b^2 + 17408b^4) + \cdots , \tag{80}$$

$$\begin{aligned} E \;=\; & \frac{1}{18}(9a^2 + 16b^2) + \frac{\epsilon}{2592}(-729a^4 - 2304a^2b^2 - 1280b^4) \;+ \\ & + \frac{\epsilon^2}{219469824}(48401955a^6 + 206098560a^4b^2 + 220662144a^2b^4 \;+ \\ & + 108867584b^6) + \cdots , \end{aligned} \tag{81}$$

$$\begin{aligned} x \;=\; & a\cos 3s + \frac{\epsilon}{224}(-56ab^2\cos 5s - 7a^3\cos 9s - 8ab^2\cos 11s) \;+ \\ & + \frac{\epsilon^2}{24837120}[-3557400a^3b^2\cos s + (5063520a^3b^2 \;+ \\ & + 3449600ab^4)\cos 5s + (460845a^5 - 98560a^3b^2)\cos 9s \;+ \\ & + (1399200a^3b^2 + 492800ab^4)\cos 11s + 500192ab^4\cos 13s \;+ \\ & + 88935a^5\cos 15s + 243672a^3b^2\cos 17s \;+ \\ & + 166880ab^4\cos 19s] + \cdots , \end{aligned} \tag{82}$$

$$\begin{aligned} y \;=\; & b\cos 4s + \frac{\epsilon}{126}(42a^2b\cos 2s - 6a^2b\cos 10s - 7b^3\cos 12s) \;+ \\ & + \frac{\epsilon^2}{41912640}[(-7858620a^4b - 4324320a^2b^3)\cos 2s \;- \\ & - 2794176a^2b^3\cos 6s - 72765a^4b\cos 8s + (1122660a^4b \;- \\ & - 237600a^2b^3)\cos 10s + (2494800a^2b^3 + 1422960b^5)\cos 12s \;+ \\ & + 251559a^4b\cos 16s + 691200a^2b^3\cos 18s \;+ \\ & + 474320b^5\cos 20s] + \cdots . \end{aligned} \tag{83}$$

These equations describe periodic pretzel orbits. A change of sign of b gives a trajectory symmetric to the original one with respect to the x-axis, whereas a change of sign of a leaves it invariant. The domain of existence of the family in the (q, E)-plane is delimited by the curves $x34$ where they degenerate into x-axial orbits and $y34$ where they degenerate into y-axial orbits (Figure 2). Curve $x34$ is given by

$$q = \frac{3}{4} + \frac{3\epsilon}{16}E - \frac{87\epsilon^2}{1792}E^2 + \cdots \tag{84}$$

and curve $y34$ by

$$q = \frac{3}{4} - \frac{3\epsilon}{16}E - \frac{129\epsilon^2}{1792}E^2 + \cdots . \tag{85}$$

5.4.2. *Family* $3:4b$

With $C = 0$ and $D = b$, we obtain for q, σ and E expressions which coincide with the corresponding expressions for family $3:4a$ up to order 5; x and y are given by

$$\begin{aligned} x \;=\; & a\cos 3s + \frac{\epsilon}{224}(56ab^2\cos 5s - 7a^3\cos 9s + 8ab^2\cos 11s) + \\ & + \frac{\epsilon^2}{24837120}[3557400a^3b^2\cos s + (-5063520a^3b^2 - \\ & - 3449600ab^4)\cos 5s + (460845a^5 - 98560a^3b^2)\cos 9s + \\ & + (-1399200a^3b^2 - 492800ab^4)\cos 11s + 500192ab^4\cos 13s + \\ & + 88935a^5\cos 15s - 243672a^3b^2\cos 17s + 166880ab^4\cos 19s] + \cdots , \end{aligned} \tag{86}$$

$$\begin{aligned} y \;=\; & b\sin 4s + \frac{\epsilon}{126}(-42a^2b\sin 2s - 6a^2b\sin 10s + 7b^3\sin 12s) + \\ & + \frac{\epsilon^2}{41912640}[(7858620a^4b + 4324320a^2b^3)\sin 2s + \\ & + 2794176a^2b^3\sin 6s + 72765a^4b\sin 8s + (1122660a^4b - \\ & - 237600a^2b^3)\sin 10s + (-2494800a^2b^3 - 1422960b^5)\sin 12s + \\ & + 251559a^4b\sin 16s - 691200a^2b^3\sin 18s + 474320b^5\sin 20s] + \cdots . \end{aligned} \tag{87}$$

These equations describe periodic antipretzel orbits. A change of sign of b reverses the direction of motion on the same trajectory, whereas a change of sign of a leaves it invariant. In the (q, E)-plane, they have the same domain of existence as the pretzel orbits and degenerate into axial orbits on the same boundaries.

5.5. RESONANCE $3:5$

We have obtained, at order 7, the same relation between C and D as in the case of resonance $3:4$. The six types of solutions can be grouped in the same manner into two families.

5.5.1. *Family* $3:5a$

With $C = b$ and $D = 0$, we obtain

$$q \;=\; \frac{3}{5} + \frac{\epsilon}{120}(9a^2 - 25b^2) + \frac{\epsilon^2}{5120}(-231a^4 + 700a^2b^2 + 625b^4) + \cdots , \tag{88}$$

$$\begin{aligned} \sigma \;=\; & \frac{1}{3} + \frac{\epsilon}{216}(-27a^2 - 50b^2) + \\ & + \frac{\epsilon^2}{82944}(7020a^4 + 22650a^2b^2 + 21875b^4) + \cdots , \end{aligned} \tag{89}$$

$$\begin{aligned} E \;=\; & \frac{1}{18}(9a^2+25b^2)+\frac{\epsilon}{2592}(-729a^4-3600a^2b^2-3125b^4)\;+ \\ & +\frac{\epsilon^2}{71663616}(15804720a^6+105538275a^4b^2+177509475a^2b^4+ \\ & +118250000b^6)+\cdots \end{aligned} \tag{90}$$

$$\begin{aligned} x \;=\; & a\cos 3s+\frac{\epsilon}{128}(-20ab^2\cos 7s-4a^3\cos 9s-5ab^2\cos 13s)\;+ \\ & +\frac{\epsilon^2}{147603456}[-20660640a^3b^2\cos s+(15915900a^3b^2\;+ \\ & +20020000ab^4)\cos 7s+(2738736a^5+1401400a^3b^2)\cos 9s\;+ \\ & +(7732725a^3b^2+5005000ab^4)\cos 13s+528528a^5\cos 15s\;+ \\ & +3374800ab^4\cos 17s+1844115a^3b^2\cos 19s\;+ \\ & +1609300ab^4\cos 23s]+\cdots, \end{aligned} \tag{91}$$

$$\begin{aligned} y \;=\; & b\cos 5s+\frac{\epsilon}{1152}(300a^2b\cos s-75a^2b\cos 11s-100b^3\cos 15s)\;+ \\ & +\frac{\epsilon^2}{3985293312}[(-583783200a^4b-797296500a^2b^3)\cos s\;+ \\ & +54054000a^4b\cos 7s-169884000a^2b^3\cos 9s+(145945800a^4b\;+ \\ & +138513375a^2b^3)\cos 11s+(308107800a^2b^3\;+ \\ & +330330000b^5)\cos 15s+35626500a^4b\cos 17s\;+ \\ & +125519625a^2b^3\cos 21s+110110000b^5\cos 25s]+\cdots. \end{aligned} \tag{92}$$

A change of sign of a or b gives a trajectory which is symmetric to the original one with respect to the x-axis or to the y-axis (both symmetry operations applied to this orbit give the same result). The domain of existence of these orbits in the (q, E)-plane is delimited by the curves $x35$ where they degenerate into x-axial orbits and $y35$ where they degenerate into y-axial orbits (Figure 2). Curve $x35$ is given by

$$q=\frac{3}{5}+\frac{3\epsilon}{20}E-\frac{3\epsilon^2}{256}E^2+\cdots \tag{93}$$

and curve $y35$ by

$$q=\frac{3}{5}-\frac{3\epsilon}{20}E-\frac{39\epsilon^2}{1280}E^2+\cdots. \tag{94}$$

5.5.2. *Family* $3:5b$

With $C = 0$ and $D = b$, we obtain for q, σ and E expressions which coincide with the corresponding expressions for family $3:5a$ up to order 6; x and y are given by

$$\begin{aligned} x \;=\; & a\cos 3s + \frac{\epsilon}{128}(20ab^2\cos 7s - 4a^3\cos 9s + 5ab^2\cos 13s) \;+ \\ & + \frac{\epsilon^2}{147603456}[20660640a^3b^2\cos s + (-15915900a^3b^2 + \\ & - 20020000ab^4)\cos 7s + (2738736a^5 + 1401400a^3b^2)\cos 9s + \\ & + (-7732725a^3b^2 - 5005000ab^4)\cos 13s + 528528a^5\cos 15s + \\ & + 3374800ab^4\cos 17s - 1844115a^3b^2\cos 19s \;+ \\ & + 1609300ab^4\cos 23s] + \cdots , \end{aligned} \tag{95}$$

$$\begin{aligned} y \;=\; & b\sin 5s + \frac{\epsilon}{1152}(-300a^2b\sin s - 75a^2b\sin 11s + 100b^3\sin 15s) + \\ & + \frac{\epsilon^2}{3985293312}[(583783200a^4b + 797296500a^2b^3)\sin s - \\ & - 54054000a^4b\sin 7s + 169884000a^2b^3\sin 9s + \\ & + (145945800a^4b + 138513375a^2b^3)\sin 11s + \\ & + (-308107800a^2b^3 - 330330000b^5)\sin 15s + \\ & + 35626500a^4b\sin 17s - 125519625a^2b^3\sin 21s + \\ & + 110110000b^5\sin 25s] + \cdots . \end{aligned} \tag{96}$$

A change of sign of a or b reverses the direction of motion on the same trajectory. These orbits have the same domain of existence as those of family $3{:}5a$ and degenerate into axial orbits on the same boundaries.

5.6. RESONANCE $4:5$

We have obtained, at order 8, the relation

$$CD(C - D)(C + D)(C^2 - 2CD - D^2)(C^2 + 2CD - D^2) = 0. \tag{97}$$

It admits eight types of solutions:

$(a1)$ $D = 0$,
$(a2)$ $C = 0$,
$(a3, a4)$ $C = \pm D$,
$(b1, b2)$ $C = (1 \pm \sqrt{2})D$,
$(b3, b4)$ $C = (-1 \pm \sqrt{2})D$.

The changes of variables

$$s = s' + \tfrac{1}{4}k\pi, \quad k = 1, 2, 3 \tag{98}$$

show that we can group them into two families.

5.6.1. *Family* $4:5a$

With $C = b$ and $D = 0$, we obtain

$$\begin{aligned} q \;=\;& \frac{4}{5} + \frac{\epsilon}{160}(16a^2 - 25b^2) + \\ &+ \frac{\epsilon^2}{737280}(-55552a^4 + 145600a^2b^2 + 10625b^4) + \cdots , \end{aligned} \tag{99}$$

$$\begin{aligned} \sigma \;=\;& \frac{1}{4} + \frac{\epsilon}{256}(-24a^2 - 25b^2) + \\ &+ \frac{5\epsilon}{294912}(3744a^4 + 5920a^2b^2 + 4375b^4) + \cdots , \end{aligned} \tag{100}$$

$$\begin{aligned} E \;=\;& \frac{1}{32}(16a^2 + 25b^2) + \frac{\epsilon}{8192}(-2304a^4 - 6400a^2b^2 - 3125b^4) + \\ &+ \frac{5\epsilon^2}{679477248}(29970432a^6 + 113260160a^4b^2 + 106365440a^2b^4 + \\ &+ 49753125b^6) + \cdots , \end{aligned} \tag{101}$$

$$\begin{aligned} x \;=\;& a\cos 4s + \frac{\epsilon}{288}(-90ab^2\cos 6s - 9a^3\cos 12s - 10ab^2\cos 14s) + \\ &+ \frac{\epsilon^2}{120766464}[-22014720a^3b^2\cos 2s + (33415200a^3b^2 + \\ &+ 18427500ab^4)\cos 6s + (2240784a^5 - 1146600a^3b^2)\cos 12s + \\ &+ (7352800a^3b^2 + 2047500ab^4)\cos 14s + 2518425ab^4\cos 16s + \\ &+ 432432a^5\cos 20s + 1104320a^3b^2\cos 22s + \\ &+ 704925ab^4\cos 24s] + \cdots , \end{aligned} \tag{102}$$

$$\begin{aligned} y \;=\;& b\cos 5s + \frac{\epsilon}{4608}(1800a^2b\cos 3s - 200a^2b\cos 13s - 225b^3\cos 15s) + \\ &+ \frac{\epsilon^2}{1932263424}[(-424569600a^4b - 127764000a^2b^3)\cos 3s - \\ &- 189189000a^2b^3\cos 7s - 9828000a^4b\cos 11s + (47174400a^4b - \\ &- 32396000a^2b^3)\cos 13s + (112694400a^2b^3 + 50675625b^5)\cos 15s + \\ &+ 10332000a^4b\cos 21s + 26429000a^2b^3\cos 23s + \\ &+ 16891875b^5\cos 25s] + \cdots . \end{aligned} \tag{103}$$

A change of sign of a gives a trajectory symmetric to the original one with respect to the y-axis, whereas a change of sign of b leaves it invariant. The domain of existence of these orbits in the (q, E)-plane is delimited by the curves $x45$ where they degenerate into x-axial orbits and $y45$ where they degenerate into y-axial orbits (Figure 2). Curve $x45$ is given by

$$q = \frac{4}{5} + \frac{\epsilon}{5}E - \frac{11\epsilon^2}{144}E^2 + \cdots \tag{104}$$

and curve $y45$ by

$$q = \frac{4}{5} - \frac{\epsilon}{5}E - \frac{73\epsilon^2}{720}E^2 + \cdots . \tag{105}$$

5.6.2. *Family* $4:5b$

With $C = \sqrt{2+\sqrt{2}}b/2$ and $D = (\sqrt{2+\sqrt{2}})(\sqrt{2}-1)b/2$ we obtain for q, σ and E expressions which coincide with the corresponding expressions for family $4:5a$ up to order 7; x and y are given by

$$\begin{aligned} x &= a\cos 4s + \frac{\epsilon}{288}[-45\sqrt{2}ab^2(\cos 6s + \sin 6s) - 9a^3\cos 12s - \\ &\quad -5\sqrt{2}ab^2(\cos 14s + \sin 14s)] + \cdots , \qquad (106)\\ y &= \frac{\sqrt{2+\sqrt{2}}}{2}[b\cos 5s + (\sqrt{2}-1)b\sin 5s] + \\ &\quad + \frac{25\sqrt{2+\sqrt{2}}\epsilon}{9216}\{72a^2b[\cos 3s - (\sqrt{2}-1)\sin 3s] - \\ &\quad - 8a^2b[\cos 13s + (\sqrt{2}-1)\sin 13s] - \\ &\quad - 9b^3[(\sqrt{2}-1)\cos 15s + \sin 15s]\} + \cdots . \qquad (107) \end{aligned}$$

A change of sign of a reverses the direction of motion on the same trajectory, whereas a change of sign of b leaves it invariant. These orbits have the same domain of existence as those of family $4:5a$ and degenerate into axial orbits on the same boundaries.

5.7. NUMERICAL DETERMINATION OF THE BOUNDARIES

In order to assess the validity of the analytical determination of the domains of existence of different families of periodic orbits, we have recomputed the boundaries with a numerical method. We use the technique of Poincaré sections. On the x-boundary of the $m:n$ family domain, the periodic orbits of the family degenerate into x-axial orbits. These orbits and their neighbours are best studied in the plane of section $x = 0$. Let T be the Poincaré application and T' its derivative at the origin (see paper I). At the point where an $m:n$ periodic orbit springs from an x-axial orbit, T'^m, the mth iteration of the Poincaré application, is marginally stable. So, on the x-boundary, we have

$$\operatorname{tr} T'^m = \pm 2 \quad \text{or} \quad \operatorname{tr} T' = 2\cos\frac{k\pi}{m}, \quad k = 0, 1, \ldots . \tag{108}$$

The precise value of k can be deduced from the case of small amplitude. In this case, x and y are given by expressions (13) and (14) and $\sigma = 1/m$. Without restricting the generality, we can suppose that $A > 0$ and $B = 0$. We obtain points of section ($x = 0$ and $\dot{x} > 0$) for $s = (2k + \frac{3}{2})\pi/m$. For two consecutive sections, corresponding to $s_0 = 3\pi/2m$ and $s_1 = 7\pi/2m$, we have

$$y_0 = C\cos\frac{3n\pi}{2m} + D\sin\frac{3n\pi}{2m}, \tag{109}$$

$$\dot{y}_0 = -\frac{n}{m}C\sin\frac{3n\pi}{2m} + \frac{n}{m}D\cos\frac{3n\pi}{2m}, \tag{110}$$

$$y_1 = C\cos\frac{7n\pi}{2m} + D\sin\frac{7n\pi}{2m}, \tag{111}$$

$$\dot{y}_1 = -\frac{n}{m}C\sin\frac{7n\pi}{2m} + \frac{n}{m}D\cos\frac{7n\pi}{2m}. \tag{112}$$

In this approximation the Poincaré application is linear and can be written

$$y_1 = y_0\cos\frac{2n\pi}{m} + \dot{y}_0\frac{m}{n}\sin\frac{2n\pi}{m}, \tag{113}$$

$$\dot{y}_1 = -y_0\frac{n}{m}\sin\frac{2n\pi}{m} + \dot{y}_0\cos\frac{2n\pi}{m}. \tag{114}$$

The x-boundary of the $m:n$ family is thus determined by the condition

$$\operatorname{tr} T' = 2\cos\frac{2n\pi}{m}. \tag{115}$$

This condition was used in our numerical computations to determine the x-boundaries.

The y-boundaries can be characterized in a similar manner. The plane of section $y = 0$ must be used to study orbits near y-axial orbits. T^n is marginally stable on the boundary, so that

$$\operatorname{tr} T' = 2\cos\frac{k\pi}{n}, \quad k = 0, 1, \ldots . \tag{116}$$

The consideration of orbits of small amplitude can give us the exact value of k. We finally obtain

$$\operatorname{tr} T' = 2\cos\frac{2m\pi}{n}. \tag{117}$$

This condition was used to compute numerically the y-boundaries.

The results of the analytical and the numerical computations can be compared in Figure 2. The analytical computation clearly lacks accuracy for the $y34$ and $y45$ boundaries and a small discrepancy is barely visible for the $x45$ boundary.

5.8. STABILITY

We have not systematically investigated the stability of the above families of periodic orbits. We have only determined the stability of the orbits represented in the bifurcation diagrams of the next section.

6. Relations with the Stability of Axial Orbits, Bifurcation Diagrams

The synthetic view of different families offered by Figure 2 helps us to understand the aspect of the bifurcation diagrams showing the families we have described originating from axial orbits. As an example, let us take $q = 0.62$. Figure 3 shows an enlargement of Figure 2 in the vicinity of $q = 0.62$.

Consider, first, the periodic orbits stemming from the x-axial orbit. When we follow the line $q = 0.62$, starting from $E = 0$, we cross successively the x-boundaries of families $3:5$, $1:2a$ and $1:2b$. Essentially the same information is displayed in the bifurcation diagram (Figure 4), where the ordinate $y_{\max}$ is the maximum value of $|y|$ reached by the orbit. This bifurcation diagram and the next one have been computed by entirely numerical methods and show the stability of the orbits.

For the same q value, starting from $E = 0$, we cross successively the y-boundaries of families $2:3$, $3:4$, $4:5$ and $1:1b$. These crossings translate into the appearance

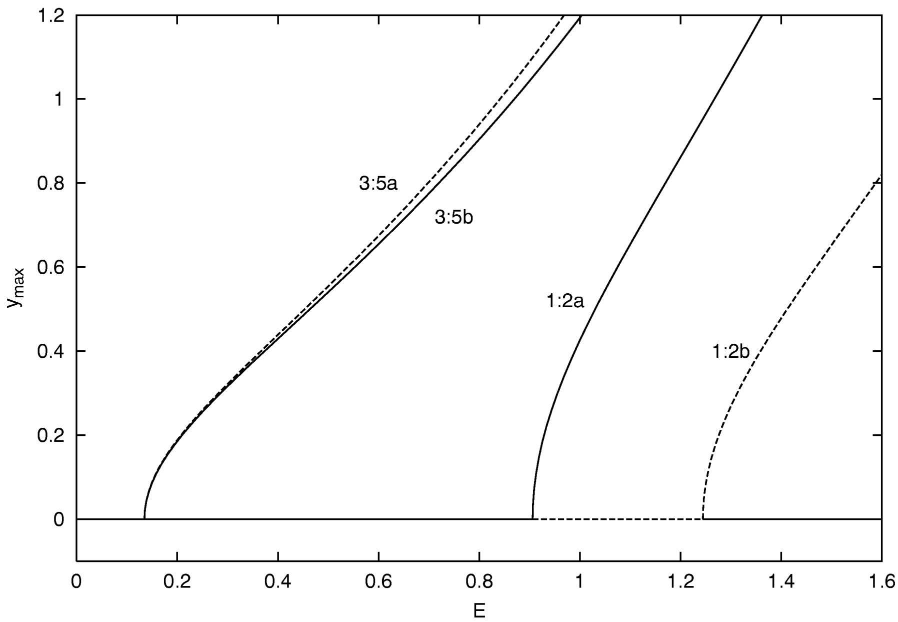

Figure 4. Bifurcation diagram showing families of periodic orbits originating from the x-axial orbit (solid line for stable orbits, dashed line for unstable ones).

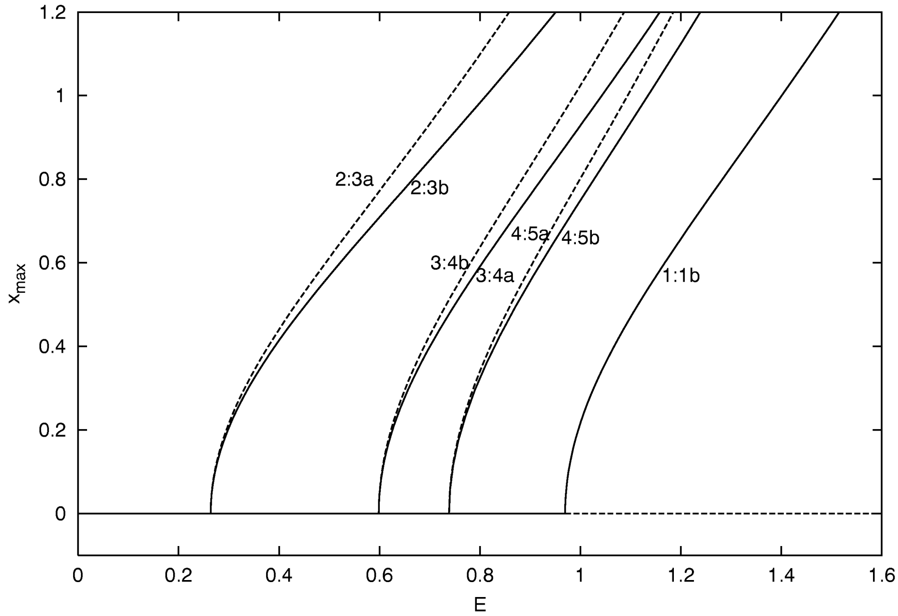

Figure 5. Bifurcation diagram showing families of periodic orbits originating from the y-axial orbit (solid line for stable orbits, dashed line for unstable ones).

of the corresponding families in the bifurcation diagram (Figure 5), where the ordinate x_{max} gives the maximum value of $|x|$ reached by the orbit.

7. Conclusion

We have demonstrated that analytical techniques based on the Lindstedt method were suitable for a quantitative study of periodic orbits. The present results were obtained in a particular potential, but the method can be used with any regular analytical potential. The algorithms we have described give the law of motion on the periodic orbits of different families up to an arbitrary order. The effective order of the series is limited only by the available computing power. The computations are rather cumbersome but they provide a complete description of the motion.

In this work, these laws of motion have been used to determine the domains of existence of different families of periodic orbits with a fairly good accuracy.

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